IV. *On an inequality of long period in the motions of the Earth and Venus.*

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In a paper "On the corrections of the elements of Delambre's Solar Tables," published in the Philosophical Transactions for 1828, I stated that the comparison of the corrections in the epochs of the sun and the sun's perigee given by late observations, with the corrections given by the observations of the last century, appeared to indicate the existence of some inequality not included in the arguments of those Tables. As soon as I had convinced myself of the necessity of seeking for some inequality of long period, I commenced an examination of the mean motions of the planets, with the view of finding one whose ratio to the mean motion of the earth could be expressed very nearly by a proportion whose terms were small: and I did not long seek in vain.

It is well known that the appearances of Venus recur in very nearly the same order every eight years: and therefore some multiple of the periodic time of Venus is nearly equal to eight years. It is easily seen that this multiple is thirteen: and consequently eight times the mean motion of Venus is nearly equal to thirteen times the mean motion of the Earth. According to Laplace, (Méc. Cél. liv. vi. chap. 6.) the mean annual motion of Venus is $650^{g}\cdot198$; that of the Earth $399^{g}\cdot993$. Hence

8 × mean annual motion of Venus	=	$5201^{g}\cdot584$
13 × mean annual motion of the Earth . .	=	5199 ·909
Difference	=	1 ·675

The difference is about $\frac{1}{240}$ of the mean annual motion of the Earth; and it implies the existence of an inequality whose period is about 240 years. No term has yet been calculated whose period is so long with respect to the periodic time of the planets disturbed *. The probability that there would be found some sensible irregularity depending on this term, may be estimated from this consideration; that in integrating the differential equations, this term receives a multiplier of $3 \times 13 \times (240)^2$, or about 2,200,000.

On the other hand, the coefficient of this term is of the fifth order (with regard to the excentricities and inclinations of the orbits). The excentricities of both orbits are small. And it is remarkable that in the present position of the perihelia, the terms which would otherwise produce a large inequality destroy each other almost exactly. The inclination however is not so small; and upon this the existing inequality depends principally for its magnitude.

The value of the principal term, calculated from the theory, I gave in a postscript to the paper above cited. I propose in the present memoir to give an account of the method of calculation, and to include other terms which are necessarily connected with the principal inequality.

PART I.

PERTURBATION OF THE EARTH'S LONGITUDE AND RADIUS VECTOR.

Section 1.

Method adopted for this investigation.

1. The motion of a disturbed planet may be represented by supposing it to move, according to the laws of undisturbed motion, in an ellipse whose dimensions and position are continually changing: the epoch of the planet's mean longitude at the origin of the time being also supposed to change. Putting a for the semi-axis major; e for the excentricity; ϖ for the longitude of perihelion; n for the mean motion in longitude in a unit of time; ε for the epoch, or the mean longitude when $t = 0$; (all which are variable): m for the mass of the planet (Venus); μ for the sum of the masses of the sun and planet; and the same letters with accents for the same quantities relative to another planet (the

* The period of the long inequality of Saturn is only about thirty times as great as the periodic time of Saturn.

Earth); the variation of the elements of the second planet's orbit will be given by the following equations:

$$\frac{d\,a'}{d\,t} = -\frac{2\,n'\,a'^2}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,\varepsilon'}$$

$$\frac{d\,n'}{d\,t} = +\frac{3\,n'^2\,a'}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,\varepsilon'}$$

$$\frac{d\,e'}{d\,t} = -\frac{n'\,a'}{\mu'\,e'}\left(1 - e'^2\right)\frac{d\,\mathrm{R}}{d\,\varepsilon'} + \frac{n'\,a'\left(1 - e'^2\right)^{\frac{1}{2}}}{\mu'\,e'}\left(\frac{d\,\mathrm{R}}{d\,\varepsilon'} + \frac{d\,\mathrm{R}}{d\,\varpi'}\right)$$

$$\frac{d\,\varpi'}{d\,t} = -\frac{n'\,a'}{\mu'\,e'}\left(1 - e'^2\right)^{\frac{1}{2}} \cdot \frac{d\,\mathrm{R}}{d\,e'}$$

$$\frac{d\,\varepsilon'}{d\,t} = -\frac{3\,n'^2\,a'}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,\varepsilon'} \cdot t + \frac{2\,n'\,a'^2}{\mu'} \cdot \frac{d\,\mathrm{R}}{d\,a'} - \frac{n'\,a'}{\mu'\,e'}\left(\left(1 - e'^2\right)^{\frac{1}{2}} - \left(1 - e'^2\right)\right)\frac{d\,\mathrm{R}}{d\,e'}$$

where R or $\dfrac{m\,(x'\,x + y'\,y)}{(x^2 + y^2 + z^2)^{\frac{3}{2}}} - \dfrac{m}{\sqrt{\{(x' - x)^2 + (y' - y)^2 + (z' - z)^2\}}}$ is expanded in terms depending on the mean motions of the two planets. These expressions are true only on the supposition that the *actual* orbit of m' is in the plane of $x\,y$, or is so little inclined that the square of the inclination may be neglected. The values of a', e', &c. on the right-hand side of the equations ought in strictness to be the true variable values. But it will in general be sufficiently accurate to put for e' the value E which it had near the time for which the investigation is made, and to consider it as constant: or at any rate the expression $\mathrm{E} + \mathrm{F}\,t$, where F is the mean value of its increase when $t = 0$: and similarly for the others. Determining thus the values of $\frac{d\,a'}{d\,t}$, $\frac{d\,e'}{d\,t}$, &c. and from them those of a', e', &c., they are to be substituted in the expressions

$$r' = a'\left\{1 + \frac{1}{2}\,e'^2 + \left(-e' + \frac{3}{8}\,e'^3 - \&c.\right)\cos\left(n't + \varepsilon' - \varpi'\right) + \left(-\frac{1}{2}\,e'^2 + \&c.\right)\cos\left(2\,n'\,t + 2\,\varepsilon' - 2\,\varpi'\right) + \&c.\right\}$$

$$v' = n't + \varepsilon' + \left(2\,e' - \frac{1}{4}\,e'^3 + \&c.\right)\sin\left(n'\,t + \varepsilon' - \varpi'\right) + \left(\frac{5}{4}\,e'^2 - \frac{11}{24}\,e'^4 + \&c.\right)\sin\left(2\,n't + 2\,\varepsilon' - 2\,\varpi'\right) + \&c.$$

and the true values of the radius vector and longitude are obtained.

2. When (as in the present instance) the inequality is so small that we may be satisfied with the principal part of it, we may in the expressions omit the powers of e'. Thus we have

$$\frac{d\,a'}{dt} = -\frac{2\,n'\,a'^2}{\mu'} \cdot \frac{d\,\text{R}}{d\,\varepsilon'}$$

$$\frac{d\,n'}{dt} = +\frac{3\,n'^2\,a'}{\mu'} \cdot \frac{d\,\text{R}}{d\,\varepsilon'}$$

$$\frac{d\,e'}{dt} = +\frac{n'\,a'}{\mu'\,e'} \cdot \frac{d\,\text{R}}{d\,\varpi'}$$

$$\frac{d\,\varpi'}{dt} = -\frac{n'\,a'}{\mu'\,e'} \cdot \frac{d\,\text{R}}{d\,e'}$$

$$\frac{d\,\varepsilon'}{dt} = -\frac{3\,n'^2\,a'}{\mu'} \cdot \frac{d\,\text{R}}{d\,\varepsilon'} \cdot t + \frac{2\,n'\,a'^2}{\mu'} \cdot \frac{d\,\text{R}}{d\,a'} - \frac{1}{2} \cdot \frac{n'\,a'\,e'}{\mu'} \cdot \frac{d\,\text{R}}{d\,e'}$$

3. Hitherto this method has been actually used (I believe) only for the calculation of secular variations. But it can be applied with great advantage in almost every case: and in the instance before us it is particularly convenient, as it requires only the development of a single term. For if in the development of R we take the terms depending on cos $\{13\,n't - 8\,n\,t + \text{A}\}$, whose coefficient is of the 5th order, it will be found that $\frac{d\,a'}{dt}$, $\frac{d\,n'}{dt}$, and $\frac{d\,\varepsilon'}{dt}$, are of the 5th order, $\frac{d\,e'}{dt}$ of the 4th order, and $\frac{d\,\varpi'}{dt}$ of the 3rd order. Integrating these expressions, and substituting them in the formula for v', there will be produced terms of the forms $\frac{p}{(13\,n' - 8\,n)^2}$ sin $\{13\,n't - 8\,n\,t + \text{B}\}$ and $\frac{q}{13\,n' - 8\,n}$ sin $\{12\,n't - 8\,n\,t + \text{C}\}$, where p is of the 5th, and q of the 4th order. And a little examination will show that no other argument will produce terms of the same or of a lower order, which are divided by the small quantity $13\,n' - 8\,n$: inasmuch as this divisor is introduced only by integration of the expressions for $\frac{d\,a'}{dt}$, &c. Our object then at present is to select in the development of R all the terms of the form A cos $\{13\,n'\,t - 8\,n\,t + \text{B}\}$. And as the inequality which we are seeking will probably be small, we may confine ourselves to those terms in which the order of the coefficient is the lowest possible: that is, to terms of the 5th order.

SECTION 2.

On the abridgement which the development admits of, and the notation which it permits us to use.

4. Let θ be the longitude of the node of the orbit of m (Venus), and ϕ its inclination: the orbit of m' (the Earth) being supposed to coincide with the plane of xy. Let v, the longitude of m, be measured* by adding the angular distance of m from its node to the longitude of the node. Then $v-\theta$ is the distance of m from the node. Let r be the true radius vector of m: then

$$x' = r' . \cos v'$$
$$y' = r' . \sin v'$$
$$x = r \{\cos(v-\theta) . \cos\theta - \sin(v-\theta) . \sin\theta . \cos\phi\}$$
$$y = r \{\cos(v-\theta) . \sin\theta + \sin(v-\theta) . \cos\theta . \cos\phi\}$$
$$z = r . \sin(v-\theta) . \sin\phi$$

Substituting these, the expression for R becomes

$$\frac{m r'}{r^2}\Big\{\cos(v'-\theta) . \cos(v-\theta) + \cos\phi . \sin(v'-\theta) . \sin(v-\theta)\Big\}$$
$$-\frac{m}{\sqrt{\Big\{r'^2 - 2r'r\Big(\cos(v'-\theta) . \cos(v-\theta) + \cos\phi . \sin(v'-\theta) . \sin(v-\theta)\Big) + r^2\Big\}}}$$

in which it must be remarked that r and v, when expressed in terms of t, will not involve the constants θ and ϕ. This may be changed into

$$\frac{m r'}{r^2}\Big\{\cos(v'-v) - \sin^2\frac{\phi}{2} . \cos(v'-v) + \sin^2\frac{\phi}{2}\cos(v'+v-2\theta)\Big\}$$
$$-\frac{m}{\sqrt{\Big\{r'^2 - 2r'r . \cos(v'-v) + r^2 + 2r'r . \sin^2\frac{\phi}{2}\cos(v'-v) - 2r'r . \sin^2\frac{\phi}{2}\cos(v'+v-2\theta)\Big\}}}$$

or,

$$\frac{m r'}{r^2}\cos(v'-v) - \frac{m}{\sqrt{\{r'^2 - 2r'r . \cos(v'-v) + r^2\}}}$$
$$+ \sin^2\frac{\phi}{2}\{\cos(v'-v) - \cos(v'+v-2\theta)\} . \Big\{\frac{m r' r}{\{r'^2 - 2r'r . \cos(v'-v) + r^2\}^{\frac{3}{2}}} - \frac{m r'}{r^2}\Big\}$$

* ϖ, the longitude of the perihelion of m, must be measured in the same manner.

$$-\frac{3}{2}\sin^4\frac{\phi}{2}\left\{\cos(v'-v)-\cos(v'+v-2\theta)\right\}^2\cdot\frac{m\,r'\,r}{\{r'^2-2\,r'r\,.\cos(v'-v)+r^2\}^{\frac{5}{2}}}$$

$+$ &c.

5. The first line of this may be expanded in the form

$$-m\left\{\frac{1}{2}\Gamma_{\frac{1}{2}}^{(0)}+\Gamma_{\frac{1}{2}}^{(1)}\cos(v'-v)+\Gamma_{\frac{1}{2}}^{(2)}\cos(2\,v'-2\,v)+\&c.\right\}$$

where $\Gamma_{\frac{1}{2}}^{(0)}$, $\Gamma_{\frac{1}{2}}^{(1)}$, &c., are functions of r' and r. We must then express r' and r in terms of $n't$ and $n\,t$, and must substitute these values in $\Gamma_{\frac{1}{2}}^{(0)}$, $\Gamma_{\frac{1}{2}}^{(1)}$, $\Gamma_{\frac{1}{2}}^{(2)}$, &c. and must express v' and v in terms of $n't$ and $n\,t$; and on multiplying the respective expressions we shall have the development necessary for our method.

6. Now upon expressing r' in terms of $n't$, the following remarkable law always holds: The index of the term of lowest order in the coefficient of such an argument as $\cos(p\,n't+\text{A})$, is p. The same is true with regard to the development of r, v', and v.

7. Now such a term as $\text{A}\cos\{13\,n't-8\,n\,t+\text{B}\}$ can be produced only by the multiplication of $\frac{\cos}{\sin}\left(k\,n't-k\,n\,t+k\,\varepsilon'-k\,\varepsilon\right)$, (from the first term in the development of $\cos\left(k\,v'-k\,v\right)$), with $\frac{\cos}{\sin}\,\overline{(13\backsim k)\;(n't+\varepsilon'-\varpi')}$ and $\frac{\cos}{\sin}\,\overline{(8\backsim k)\;(n\,t+\varepsilon-\varpi)}$ (occurring in the development of $k\,v'-k\,v$, or of $\Gamma_{\frac{1}{2}}^{(k)}$). The largest term in the coefficient, according to the rule just explained, will be of the order whose index is the sum of $13\backsim k$ and $8\backsim k$. Now if k be <8, as for instance if k be 7, the index of the order is $6+1=7$, or the term is of the 7th order, and therefore is to be rejected. And if k be >13, as for instance if $k=14$, the index of the order is $1+6=7$, and the term is to be rejected. But if k be 8, or 13, or any number between them, as for instance 10, then the order of the term is $3+2=5$, and the term is to be kept. It appears therefore that the only terms which we shall have occasion to develope, are $\Gamma_{\frac{1}{2}}^{(8)}\,.\cos(8\,v'-8\,v)$, $\Gamma_{\frac{1}{2}}^{(9)}\,.\cos(9\,v'-9\,v)$, &c. as far as $\Gamma_{\frac{1}{2}}^{(13)}\,.\cos(13\,v'-13\,v)$ inclusively.

8. Supposing then k to be not less than 8 nor greater than 13, the term $\frac{\cos}{\sin}\left(k n't - k n t + k \varepsilon' - k \varepsilon\right)$ must be multiplied by $\frac{\cos}{\sin}\left(\overline{(13-k)(n't + \varepsilon' - \varpi')} + \overline{(k-8)(n t + \varepsilon - \varpi)}\right)$ in order to produce a term of the form $A \cos (13\, n't - 8\, n t + B)$ whose coefficient is of the 5th order. The latter factor must have arisen from the product of two such terms as $e'^{\,13-k} \cdot \frac{\cos}{\sin} \overline{(13-k)(n't + \varepsilon' - \varpi')}$ and $e^{k-8} \cdot \frac{\cos}{\sin} \overline{(k-8)(n t + \varepsilon - \varpi)}$. The expansion of such a product will always produce two terms, one of which has for argument the sum of the arguments of the factors, and the other has the difference of the same arguments. The point to which I wish particularly to call the attention of the reader is this: The term of the product depending on the *sum* of the arguments is the only one which is useful to us. For instance; the product of $e'^2 . \sin 2 (n't + \varepsilon' - \varpi')$ and $e^3 . \sin 3 (nt + \varepsilon - \varpi)$ will be $- \frac{1}{2} e'^2 e^3 . \cos (2n't + 3\, n t + 2\varepsilon' + 3\varepsilon - 2\varpi' - 3\varpi) + \frac{1}{2} e'^2 e^3 . \cos (2\, n't - 3\, n t + 2\, \varepsilon' - 3\varepsilon - 2\, \varpi' + 3\, \varpi)$; the combination of the first term with $\cos (11\, n't - 11\, n t + 11\, \varepsilon' - 11\, \varepsilon)$ will produce a term of the form $A \cos (13\, n't - 8\, n t + B)$ whose coefficient is of the 5th order: the second term will not produce a term of that form. We might choose terms, as $e' . \sin (n't + \varepsilon' - \varpi')$ and $e^6 . \sin 6 (n t + \varepsilon - \varpi)$ such that the part of the product depending on the difference of the arguments, or $\frac{1}{2} e' e^6 . \cos (n't - 6\, n t + \varepsilon' - 6\varepsilon - \varpi' + 6\, \varpi)$ combining with such a term as $\cos (14\, n't - 14\, n t + 14\, \varepsilon' - 14\, \varepsilon)$, would produce a term of the form required: but its coefficient would not be of the 5th order. It is equally necessary to remark that, in multiplying the term thus selected by $\frac{\cos}{\sin}\left(k n't - k n t + k \varepsilon' - k \varepsilon\right)$, we again preserve only that part of the product depending on the *sum* of the arguments.

9. On the circumstance that, in taking the product of two circular functions, we have to retain only the term whose argument is the sum of the arguments, depends the principle of our notation. For whenever (in an advanced stage of the operations) such a term as $\frac{\cos}{\sin}\left(2\, n't + 3\, n t + 2\, \varepsilon' + 3\, \varepsilon - 2\, \varpi' - 3\, \varpi\right)$ occurs, we shall know that, being formed in accordance with this rule, it must

have arisen from the product of $e'^2 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(2n't+2\varepsilon'-2\varpi'\right)$ and $e^3 \begin{smallmatrix}\cos\\ \sin\end{smallmatrix}\left(3nt+3\varepsilon-3\varpi\right)$; its coefficient therefore can only be $e'^2 e^3$. And conversely, from seeing this coefficient, we should be certain that the argument would be $2(n't+\varepsilon'-\varpi') + 3(nt+\varepsilon-\varpi)$. Instead therefore of writing

$$e'^2 e^3 . \cos(2n't + 3nt + 2\varepsilon' + 3\varepsilon - 2\varpi' - 3\varpi)$$

we might simply write

$$e'^2 e^3 . \cos$$

omitting the argument entirely. But it will be found more convenient to retain the figures in the argument, writing it thus,

$$e'^2 e^3 . \cos(2+3)$$

the first figure being always appropriated to the accented argument. And when this term is multiplied by $\cos(11n't - 11nt + 11\varepsilon' - 11\varepsilon)$ or $\cos(11-11)$, we may write down the result

$$\tfrac{1}{2} e'^2 e^3 . \cos(13-8)$$

without any fear of mistake. For we know that the argument must have been produced by adding $2(n't+\varepsilon'-\varpi')$, $3(nt+\varepsilon-\varpi)$, and $11(n't-nt+\varepsilon'-\varepsilon)$, and thus when a result is obtained the term can be filled up.

10. If we examine the second line in the last expression of (4), it is easily seen that $\sin^2\frac{\phi}{2}$, a quantity of the second order (considering $\sin\frac{\phi}{2}$ as of the same order with e' and e) enters as multiplier into two terms: of which the first, or $\sin^2\frac{\phi}{2} . \cos(v'-v)$, when developed will have in every term one part of the argument produced by a subtraction; and therefore, when combined with the expansion of the term multiplying it, will produce terms $\cos(13-8)$ of the 7th order at lowest; the first term therefore is useless. But the second, or $-\sin^2\frac{\phi}{2} . \cos(v'+v-2\theta)$, is exactly analogous to $e^2\cos(v'+v-2\varpi)$, which would arise from the product of $e^2\cos(2v-2\varpi)$ and $\cos(v'-v)$, and to which all the preceding remarks would apply; and examination would show that in the development of this term, in which products of $\sin^2\frac{\phi}{2}$ with powers of e' and e

will occur, the same rule must be followed, namely, that the only useful terms in the products are those in which the arguments are added. And whenever $\sin^2 \frac{\phi}{2}$ occurs in the coefficient, $-2\,\theta$ occurs in the argument; so that there will be no possibility of mistake in using the notation described in (9).

11. On examining the third line in the last expression of (4), it will be seen in the same manner that the only part of $-\frac{3}{2}\sin^4\frac{\phi}{2}\left\{\cos(v'-v)-\cos(v'+v-2\,\theta)\right\}^2$ to be preserved is $-\frac{3}{4}\sin^4\frac{\phi}{2}\,.\cos(2\,v'+2\,v-4\,\theta)$. The same remarks apply to this term as to the last; and for a similar reason the notation of (9) may be used without fear of mistake.

12. By the use of this notation we may in some instances materially shorten our expressions. For instance, we might have the terms

$$\mathrm{F}\,e'\,e^4\,.\cos(n't+4\,n\,t+\varepsilon'+4\,\varepsilon-\varpi'-4\varpi)$$
$$+\mathrm{G}\,e'\,e^2\sin^2\frac{\phi}{2}\,.\cos(n't+4\,n\,t+\varepsilon'+4\,\varepsilon-\varpi'-2\,\varpi-2\,\theta)$$
$$+\mathrm{H}\,e'\sin^4\frac{\phi}{2}\,.\cos(n't+4\,n\,t+\varepsilon'+4\,\varepsilon-\varpi'-4\,\theta)$$

All this would be expressed without the possibility of mistake by the following term,

$$\left(\mathrm{F}\,e'\,e^4+\mathrm{G}\,e'\,e^2\sin^2\frac{\phi}{2}+\mathrm{H}\,e'\sin^4\frac{\phi}{2}\right).\cos(1+4).$$

The utility of such abridgments, and the quantity of disgusting labour which they spare, can be conceived only by those who have gone through the drudgery of performing the actual operation.

13. It is only necessary to add that when we have, for the coefficient of a cosine or sine, a series proceeding by powers of e', e, $\sin^2\frac{\phi}{2}$, &c. we may always neglect all after the lowest power. For instance, the correct expression for v is

$$n\,t+\varepsilon$$
$$+\left(2\,e-\frac{1}{4}\,e^3+\frac{5}{96}\,e^5-\&c.\right)\sin(n\,t+\varepsilon-\varpi)$$

$$+\left(\frac{5}{4}e^2 - \frac{11}{24}e^4 + \&c.\right)\sin(2nt + 2\varepsilon - 2\varpi)$$

$$+\left(\frac{13}{12}e^3 - \frac{43}{64}e^5 + \&c.\right)\sin(3nt + 3\varepsilon - 3\varpi)$$

$$+\left(\frac{103}{96}e^4 - \&c.\right)\sin(4nt + 4\varepsilon - 4\varpi)$$

$$+\left(\frac{1097}{960}e^5 - \&c.\right)\sin(5nt + 5\varepsilon - 5\varpi)$$

$$+ \&c.$$

but for our purposes it will be sufficient to take $v = (0+1) + 2e \,.\, \sin(0+1) + \frac{5}{4}e^2 \,.\, \sin(0+2) + \frac{13}{12}e^3 \,.\, \sin(0+3) + \frac{103}{96}e^4 \,.\, \sin(0+4) + \frac{1097}{960}e^5 \,.\, \sin(0+5)$. For none of the terms can be of any use to us till they are multiplied, so that the largest term of the coefficient is of the 5th order; and then all the other parts will be of a higher order.

14. Putting f for $\sin\frac{\phi}{2}$, it will be seen that (in conformity with the remarks in this section), the terms of R to be developed are

$$-\frac{m}{\sqrt{\{r'^2 - 2r'r \,.\, \cos(v'-v) + r^2\}}}$$

$$-f^2\,\frac{m r' r \,.\, \cos(v' + v - 2\theta)}{\{r'^2 - 2r'r \,.\, \cos(v'-v) + r^2\}^{\frac{3}{2}}}$$

$$-\frac{3}{4}f^4 \,.\, \frac{m r'^2 r^2 \,.\, \cos(2v' + 2v - 4\theta)}{\{r'^2 - 2r'r \,.\, \cos(v'-v) + r^2\}^{\frac{5}{2}}}$$

SECTION 3.

Expansion of $\cos(kv' - kv)$, *to the fifth order.*

15. By (13) the value of $kv' - kv$ is

$$(k-k)$$

$$+ 2ke' \,.\, \sin(1+0) - 2ke \,.\, \sin(0+1) \quad \ldots\ldots\ldots \quad (A)$$

$$+ \frac{5}{4}ke'^2 \,.\, \sin(2+0) - \frac{5}{4}ke^2 \,.\, \sin(0+2) \quad \ldots\ldots \quad (B)$$

$$+\frac{13}{12}k\,e'^3\,.\sin(3+0)-\frac{13}{12}k\,e^3\,.\sin(0+3)\;\ldots\ldots\;\text{(C)}$$

$$+\frac{103}{96}k\,e'^4\,.\sin(4+0)-\frac{103}{96}k\,e^4\,.\sin(0+4)\;\ldots\ldots\;\text{(D)}$$

$$+\frac{1097}{960}k\,e'^5\,.\sin(5+0)-\frac{1097}{960}k\,e^5\,.\sin(0+5)\;\ldots\ldots\;\text{(E)}$$

The cosine is

$$\cos(k-k)\,.\cos(A+B+C+D+E)$$
$$-\sin(k-k)\,.\sin(A+B+C+D+E)$$

or

$$\cos(k-k)\,.\left\{1-\frac{A^2+2AB+B^2+2AC+2AD+2BC}{2}+\frac{A^4+4A^3B}{24}\right\}$$
$$-\sin(k-k)\,.\left\{A+B+C+D+E-\frac{A^3+3A^2B+3A^2C+3AB^2}{6}+\frac{A^5}{120}\right\}$$

omitting all products of an order above the fifth.

16. In expanding the powers of A, B, &c., and in multiplying the expansions by $\cos(k-k)$ and $\sin(k-k)$, the rules of (8) must be strictly followed. Thus we find at length for the value of $\cos(kv'-kv)$:

Principal term,

$$\cos(k-k)$$

Terms of the first order,

$$+k\,e'\,.\cos(\overline{k+1}-k)-k\,e\,.\cos(k-\overline{k-1})$$

Terms of the second order,

$$\left(\frac{1}{2}k^2+\frac{5}{8}k\right)e'^2\,.\cos\left(\overline{k+2}-k\right)-k^2\,e'\,e\,.\cos\left(\overline{k+1}-\overline{k-1}\right)$$
$$+\left(\frac{1}{2}k^2-\frac{5}{8}k\right)e^2\,.\cos\left(k-\overline{k-2}\right)$$

Terms of the third order,

$$\left(\frac{1}{6}k^3+\frac{5}{8}k^2+\frac{13}{24}k\right)e'^3.\cos\left(\overline{k+3}-k\right)+\left(-\frac{1}{2}k^3-\frac{5}{8}k^2\right)e'^2e.\cos\left(\overline{k+2}-\overline{k-1}\right)$$

$$+\left(\frac{1}{2}k^3-\frac{5}{8}k^2\right)e'e^2\,.\cos\left(\overline{k+1}-\overline{k-2}\right)$$

$$+\left(-\frac{1}{6}k^3+\frac{5}{8}k^2-\frac{13}{24}k\right)e^3\,.\cos\left(k-\overline{k-3}\right)$$

Terms of the fourth order,

$$\left(\frac{1}{24}k^4+\frac{5}{16}k^3+\frac{283}{384}k^2+\frac{103}{192}k\right)e'^4\,.\cos\left(\overline{k+4}-k\right)$$

$$+\left(-\frac{1}{6}k^4-\frac{5}{8}k^3-\frac{13}{24}k^2\right)e'^3e\,.\cos\left(\overline{k+3}-\overline{k-1}\right)$$

$$+\left(\frac{1}{4}k^4-\frac{25}{64}k^2\right)e'^2e^2\,.\cos\left(\overline{k+2}-\overline{k-2}\right)$$

$$+\left(-\frac{1}{6}k^4+\frac{5}{8}k^3-\frac{13}{24}k^2\right)e'e^3\,.\cos\left(\overline{k+1}-\overline{k-3}\right)$$

$$+\left(\frac{1}{24}k^4-\frac{5}{16}k^3+\frac{283}{384}k^2-\frac{103}{192}k\right)e^4\,.\cos\left(k-\overline{k-4}\right)$$

Terms of the fifth order,

$$\left(\frac{1}{120}k^5+\frac{5}{48}k^4+\frac{179}{384}k^3+\frac{7}{8}k^2+\frac{1097}{1920}k\right)e'^5\cos\left(\overline{k+5}-k\right)$$

$$+\left(-\frac{1}{24}k^5-\frac{5}{16}k^4-\frac{283}{384}k^3-\frac{103}{192}k^2\right)e'^4e\,.\cos\left(\overline{k+4}-\overline{k-1}\right)$$

$$+\left(\frac{1}{12}k^5+\frac{5}{24}k^4-\frac{23}{192}k^3-\frac{65}{192}k^2\right)e'^3e^2\,.\cos\left(\overline{k+3}-\overline{k-2}\right)$$

$$+\left(-\frac{1}{12}k^5+\frac{5}{24}k^4+\frac{23}{192}k^3-\frac{65}{192}k^2\right)e'^2e^3\,.\cos\left(\overline{k+2}-\overline{k-3}\right)$$

$$+\left(\frac{1}{24}k^5-\frac{5}{16}k^4+\frac{283}{384}k^3-\frac{103}{192}k^2\right)e'e^4\,.\cos\left(\overline{k+1}-\overline{k-4}\right)$$

$$+\left(-\frac{1}{120}k^5+\frac{5}{48}k^4-\frac{179}{384}k^3+\frac{7}{8}k^2-\frac{1097}{1920}k\right)e^5\,.\cos\left(k-\overline{k-5}\right)$$

This development includes every argument whose coefficient is of an order not exceeding the fifth. The coefficients however here exhibited are only the first terms of the series which represent the complete coefficients.

Section 4.

Expansion of $-\Gamma_{\frac{1}{2}}^{(k)}$, *to the fifth order.*

17. We suppose $-\dfrac{m}{\sqrt{\{r'^2 - 2r'r \cdot \cos(v'-v) + r^2\}}}$, the first term in the expression of (14), to be expanded in the form

$$-\frac{1}{2} m \Gamma_{\frac{1}{2}}^{(0)} - m\Gamma_{\frac{1}{2}}^{(1)} \cdot \cos(v'-v) - m\Gamma_{\frac{1}{2}}^{(2)} \cdot \cos(2v'-2v) - \&c.$$

$$-m\Gamma_{\frac{1}{2}}^{(k)} \cdot \cos(kv'-kv) - \&c.$$

where $\Gamma_{\frac{1}{2}}^{(0)}$, $\Gamma_{\frac{1}{2}}^{(1)}$, &c. are functions of r' and r only.

Let $\dfrac{1}{\sqrt{\{a'^2 - 2a'a\cos(v'-v) + a^2\}}}$

$$= \frac{1}{2} C_{\frac{1}{2}}^{(0)} + C_{\frac{1}{2}}^{(1)} \cdot \cos(v'-v) + C_{\frac{1}{2}}^{(2)} \cdot \cos(2v'-2v) + \&c. + C_{\frac{1}{2}}^{(k)} \cdot \cos(kv'-kv) + \&c.$$

then $\Gamma_{\frac{1}{2}}^{(k)}$ is the same function of r' and r that $C_{\frac{1}{2}}^{(k)}$ is of a' and a. Consequently, if $r' = a'(1+q')$, $r = a(1+q)$: and if for convenience we use the notation

$$(m, n)\, C_{\frac{1}{2}}^{(k)}$$

to express that which is commonly written

$$a'^m \cdot a^n \cdot \frac{d^{m+n} C_{\frac{1}{2}}^{(k)}}{d\,a'^m \cdot d\,a^n}$$

we shall have for $-\Gamma_{\frac{1}{2}}^{(k)}$ the following expression:

$$-C_{\frac{1}{2}}^{(k)} - (1,0)C_{\frac{1}{2}}^{(k)} \cdot q' - (2,0)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^2}{2} - (3,0)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^3}{6} - (4,0)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^4}{24} - (5,0)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^5}{120}$$

$$-(0,1)C_{\frac{1}{2}}^{(k)} \cdot q - (1,1)C_{\frac{1}{2}}^{(k)} \cdot q'q - (2,1)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^2 q}{2} - (3,1)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^3 q}{6} - (4,1)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^4 q}{24}$$

$$-(0,2)C_{\frac{1}{2}}^{(k)} \cdot \frac{q^2}{2} - (1,2)C_{\frac{1}{2}}^{(k)} \cdot \frac{q' q^2}{2} - (2,2)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^2 q^2}{4} - (3,2)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^3 q^2}{12}$$

$$-(0,3)C_{\frac{1}{2}}^{(k)} \cdot \frac{q^3}{6} - (1,3)C_{\frac{1}{2}}^{(k)} \cdot \frac{q' q^3}{6} - (2,3)C_{\frac{1}{2}}^{(k)} \cdot \frac{q'^2 q^3}{12}$$

$$-(0,4)C_{\frac{1}{2}}^{(k)} \cdot \frac{q^4}{24} - (1,4)C_{\frac{1}{2}}^{(k)} \cdot \frac{q' q^4}{24}$$

$$-(0,5)C_{\frac{1}{2}}^{(k)} \cdot \frac{q^5}{120}$$

18. The value of r, contracted according to the system of (13), is

$$a\left\{1-e.\cos(0+1)-\frac{1}{2}e^2.\cos(0+2)-\frac{3}{8}e^3.\cos(0+3)-\frac{1}{3}e^4.\cos(0+4)\right.$$
$$\left.-\frac{125}{384}e^5.\cos(0+5)\right\}$$

whence $q=$

$$-e\cos(0+1)-\frac{1}{2}e^2.\cos(0+2)-\frac{3}{8}e^3.\cos(0+3)-\frac{1}{3}e^4.\cos(0+4)$$
$$-\frac{125}{384}e^5.\cos(0+5)$$

and a similar expression holds for q'. Substituting these in the expression above, and following strictly the precept of (8), we find for the development of $-\Gamma_{\frac{1}{2}}^{(k)}$,

Principal term,

$$-C_{\frac{1}{2}}^{(k)}$$

Terms of the first order,

$$+(1,0)\,C_{\frac{1}{2}}^{(k)}.e'\cos(1+0)+(0,1)\,C_{\frac{1}{2}}^{(k)}.e\cos(0+1)$$

Terms of the second order,*

$$\left\{\frac{1}{2}(1,0)-\frac{1}{4}(2,0)\right\}C_{\frac{1}{2}}^{(k)}.e'^2\cos(2+0)-\frac{1}{2}(1,1)\,C_{\frac{1}{2}}^{(k)}.e'e\cos(1+1)$$
$$+\left\{\frac{1}{2}(0,1)-\frac{1}{4}(0,2)\right\}C_{\frac{1}{2}}^{(k)}.e^2\cos(0+2)$$

Terms of the third order,

$$\left\{\frac{3}{8}(1,0)-\frac{1}{4}(2,0)+\frac{1}{24}(3,0)\right\}C_{\frac{1}{2}}^{(k)}.e'^3\cos(3+0)$$
$$+\left\{-\frac{1}{4}(1,1)+\frac{1}{8}(2,1)\right\}C_{\frac{1}{2}}^{(k)}.e'^2e\cos(2+1)$$

* In this and the succeeding expressions, when a cosine is multiplied by the sum of several differential coefficients of $C_{\frac{1}{2}}^{(k)}$, the symbols of differentiation are bracketed together, and $C_{\frac{1}{2}}^{(k)}$ is put at the end of the bracket.

$$+\left\{-\frac{1}{4}(1,1)+\frac{1}{8}(1,2)\right\}C_{\frac{1}{2}}^{(k)}.\,e'e^2\cos(1+2)$$

$$+\left\{\frac{3}{8}(0,1)-\frac{1}{4}(0,2)+\frac{1}{24}(0,3)\right\}C_{\frac{1}{2}}^{(k)}.\,e^3\cos(0+3)$$

Terms of the fourth order,

$$\left\{\frac{1}{3}(1,0)-\frac{1}{4}(2,0)+\frac{1}{16}(3,0)-\frac{1}{192}(4,0)\right\}C_{\frac{1}{2}}^{(k)}.\,e'^4\cos(4+0)$$

$$+\left\{-\frac{3}{16}(1,1)+\frac{1}{8}(2,1)-\frac{1}{48}(3,1)\right\}C_{\frac{1}{2}}^{(k)}.\,e'^3e\cos(3+1)$$

$$+\left\{-\frac{1}{8}(1,1)+\frac{1}{16}(2,1)+\frac{1}{16}(1,2)-\frac{1}{32}(2,2)\right\}C_{\frac{1}{2}}^{(k)}.\,e'^2e^2\cos(2+2)$$

$$+\left\{-\frac{3}{16}(1,1)+\frac{1}{8}(1,2)-\frac{1}{48}(1,3)\right\}C_{\frac{1}{2}}^{(k)}.\,e'e^3\cos(1+3)$$

$$+\left\{\frac{1}{3}(0,1)-\frac{1}{4}(0,2)+\frac{1}{16}(0,3)-\frac{1}{192}(0,4)\right\}C_{\frac{1}{2}}^{(k)}.\,e^4\cos(0+4)$$

Terms of the fifth order,

$$\left\{\frac{125}{384}(1,0)-\frac{25}{96}(2,0)+\frac{5}{64}(3,0)-\frac{1}{96}(4,0)+\frac{1}{1920}(5,0)\right\}C_{\frac{1}{2}}^{(k)}.\,e'^5\cos(5+0)$$

$$+\left\{-\frac{1}{6}(1,1)+\frac{1}{8}(2,1)-\frac{1}{32}(3,1)+\frac{1}{384}(4,1)\right\}C_{\frac{1}{2}}^{(k)}.\,e'^4e\cos(4+1)$$

$$+\left\{-\frac{3}{32}(1,1)+\frac{1}{16}(2,1)+\frac{3}{64}(1,2)-\frac{1}{96}(3,1)-\frac{1}{32}(2,2)\right.$$
$$\left.+\frac{1}{192}(3,2)\right\}C_{\frac{1}{2}}^{(k)}.\,e'^3e^2\cos(3+2)$$

$$+\left\{-\frac{3}{32}(1,1)+\frac{3}{64}(2,1)+\frac{1}{16}(1,2)-\frac{1}{32}(2,2)-\frac{1}{96}(1,3)\right.$$
$$\left.+\frac{1}{192}(2,3)\right\}C_{\frac{1}{2}}^{(k)}.\,e'^2e^3\cos(2+3)$$

$$+\left\{-\frac{1}{6}(1,1)+\frac{1}{8}(1,2)-\frac{1}{32}(1,3)+\frac{1}{384}(1,4)\right\}C_{\frac{1}{2}}^{(k)}.\,e'e^4\cos(1+4)$$

$$+\left\{\frac{125}{384}(0,1)-\frac{25}{96}(0,2)+\frac{5}{64}(0,3)-\frac{1}{96}(0,4)+\frac{1}{1920}(0,5)\right\}C_{\frac{1}{2}}^{(k)}.\,e^5\cos(0+5)$$

Every argument is included whose coefficient is of an order not superior to the fifth: but only the lowest order of each coefficient is taken.

SECTION 5.

Selection of the coefficients of cos (13 − 8) *in the development of*

$$-\frac{m}{\sqrt{\{r'^2 - 2r'r \cdot \cos(v'-v) + r^2\}}}$$

19. For this purpose, as the general term in the expansion of $-\frac{m}{\sqrt{\{r'^2 - 2r'r \cdot \cos(v'-v) + r^2\}}}$ is $-m\,\Gamma_{\frac{1}{2}}^{(k)} \cdot \cos(kv' - kv)$, we ought to multiply together the expressions of (16) and (18), to multiply the product by m, and then giving different values to k to select those terms which have for argument (13 − 8). But without going through this labour we may, when a value is assumed for k, select by the eye the terms required. As we have explained in (7), the values which it is proper to give to k are 8, 9, 10, 11, 12, 13.

20. Thus we obtain the following coefficients of cos (13 − 8):

$$k = 8.$$

$$m \times \left\{ -\frac{239753}{240}(0,0)^{*} + \frac{178109}{768}(1,0) - \frac{4217}{192}(2,0) + \frac{407}{384}(3,0) - \frac{5}{192}(4,0) + \frac{1}{3840}(5,0) \right\} C_{\frac{1}{2}}^{(8)} \cdot e'^5 \quad \ldots \quad (L^{(8)} \cdot e'^5)$$

$$k = 9.$$

$$m \times \left\{ \frac{1955097}{384}(0,0) - \frac{88029}{96}(1,0) + \frac{217233}{768}(0,1) + \frac{4041}{64}(2,0) - \frac{9781}{192}(1,1) - \frac{189}{96}(3,0) + \frac{449}{128}(2,1) + \frac{3}{128}(4,0) - \frac{7}{64}(3,1) + \frac{1}{768}(4,1) \right\} C_{\frac{1}{2}}^{(9)} \cdot e'^4 e \quad \ldots \quad (L^{(9)} \cdot e'^4 e)$$

* By $(0,0)\, C_{\frac{1}{2}}^{(8)}$ is meant the same as $C_{\frac{1}{2}}^{(8)}$.

$k = 10.$

$$m \times \left\{ -\frac{492625}{48}(0,0) + \frac{86275}{64}(1,0) - \frac{53485}{48}(0,1) - \frac{1925}{32}(2,0) \right.$$

$$+ \frac{9367}{64}(1,1) - \frac{2815}{96}(0,2) + \frac{175}{192}(3,0) - \frac{209}{32}(2,1) + \frac{493}{128}(1,2)$$

$$\left. + \frac{19}{192}(3,1) - \frac{11}{64}(2,2) + \frac{1}{384}(3,2) \right\} C_{\frac{1}{2}}^{(10)} . e'^3 e^2 \quad . \; . \; . \quad (L^{(10)} . e'^3 e^2)$$

$k = 11.$

$$m \times \left\{ \frac{492107}{48}(0,0) - \frac{20999}{24}(1,0) + \frac{52283}{32}(0,1) + \frac{913}{48}(2,0) \right.$$

$$- \frac{2231}{16}(1,1) + \frac{2695}{32}(0,2) + \frac{97}{32}(2,1) - \frac{115}{16}(1,2) + \frac{539}{384}(0,3)$$

$$\left. + \frac{5}{32}(2,2) - \frac{23}{192}(1,3) + \frac{1}{384}(2,3) \right\} C_{\frac{1}{2}}^{(11)} . e'^2 e^3 \quad . \; . \; . \quad (L^{(11)} . e'^2 e^3)$$

$k = 12.$

$$m \times \left\{ -\frac{20337}{4}(0,0) + \frac{6779}{32}(1,0) - \frac{2117}{2}(0,1) + \frac{2119}{48}(1,1) \right.$$

$$- \frac{321}{4}(0,2) + \frac{107}{32}(1,2) - \frac{21}{8}(0,3) + \frac{7}{64}(1,3) - \frac{1}{32}(0,4)$$

$$\left. + \frac{1}{768}(1,4) \right\} C_{\frac{1}{2}}^{(12)} . e' e^4 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (L^{(12)} . e' e^4)$$

$k = 13.$

$$m \times \left\{ \frac{240643}{240}(0,0) + \frac{24571}{96}(0,1) + \frac{1219}{48}(0,2) + \frac{235}{192}(0,3) \right.$$

$$\left. + \frac{11}{384}(0,4) + \frac{1}{3840}(0,5) \right\} C_{\frac{1}{2}}^{(13)} . e^5 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (L^{(13)} . e^5)$$

The arguments of the cosines multiplied respectively by these coefficients, it must be recollected, are not similar. Their form will be determined by the considerations mentioned in (10).

21. The next term of R to be developed, by (14), is

$$-m \cdot \frac{r' r}{\{r'^2 - 2 r' r \cdot \cos(v' - v) + r^2\}^{\frac{3}{2}}} \cdot f^2 \cdot \cos(v' + v - 2\theta)$$

We shall put $\Gamma_{\frac{3}{2}}^{(k)}$ for the general term in the expansion

$$\frac{r' r}{\{r'^2 - 2 r' r \cdot \cos(v' - v) + r^2\}^{\frac{3}{2}}} = \tfrac{1}{2}\Gamma_{\frac{3}{2}}^{(0)} + \Gamma_{\frac{3}{2}}^{(1)} \cdot \cos(v' - v) + \Gamma_{\frac{3}{2}}^{(2)} \cdot \cos(2v' - 2v) + \&c.;$$

And $C_{\frac{3}{2}}^{(k)}$ for the general term in the expansion

$$\frac{a' a}{\{a'^2 - 2 a' a \cdot \cos(v' - v) + a^2\}^{\frac{3}{2}}} = \tfrac{1}{2} C_{\frac{3}{2}}^{(0)} + C_{\frac{3}{2}}^{(1)} \cos(v' - v) + C_{\frac{3}{2}}^{(2)} \cos(2v' - 2v) + \&c.$$

Section 6.

Development of $f^2 \cdot \cos(v' + v - 2\theta)$, to the fifth order.

22. As the multiplier f^2 is of the second order, we want $\cos(v' + v - 2\theta)$ only to the third order. Now, by (13), $v' + v - 2\theta =$

$$(1 + 1) - 2\theta$$

$$+ 2e' \sin(1 + 0) + 2e \cdot \sin(0 + 1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

$$+ \frac{5}{4} e'^2 \cdot \sin(2 + 0) + \frac{5}{4} e^2 \cdot \sin(0 + 2) \quad . \quad . \quad . \quad . \quad . \quad (B)$$

$$+ \frac{13}{12} e'^3 \sin(3 + 0) + \frac{13}{12} e^3 \cdot \sin(0 + 3) \quad . \quad . \quad . \quad . \quad . \quad (C)$$

Its cosine, as in (15), is

$$\cos(1 + 1 - 2\theta) \cdot \left\{1 - \frac{A^2 + 2AB}{2}\right\} - \sin(1 + 1 - 2\theta) \cdot \left\{A + B + C - \frac{A^3}{6}\right\}$$

Following the rule of (8) in the expansion, we find for the value of $\cos(v' + v - 2\theta)$.

Principal Term,

$$\cos(1 + 1 - 2\theta)$$

Terms of the first order,

$$+ e' . \cos(2 + 1 - 2\theta) + e . \cos(1 + 2 - 2\theta)$$

Terms of the second order,

$$+ \frac{9}{8} e'^2 . \cos(3 + 1 - 2\theta) + e' e . \cos(2 + 2 - 2\theta) + \frac{9}{8} e^2 . \cos(1 + 3 - 2\theta)$$

Terms of the third order,

$$+ \frac{4}{3} e'^3 . \cos(4 + 1 - 2\theta) + \frac{9}{8} e'^2 e . \cos(3 + 2 - 2\theta)$$

$$+ \frac{9}{8} e' e^2 . \cos(2 + 3 - 2\theta) + \frac{4}{3} e^3 . \cos(1 + 4 - 2\theta)$$

On multiplying this by f^2 it will readily be seen that f^2 in the coefficient is always accompanied by $- 2\theta$ in the argument, and that there is a necessary connexion between them. We may therefore omit 2θ; and thus we have for the development of $f^2 . \cos(v' + v - 2\theta)$

Term of the second order,

$$f^2 . \cos(1 + 1).$$

Terms of the third order,

$$+ e' f^2 . \cos(2 + 1) + e f^2 . \cos(1 + 2).$$

Terms of the fourth order,

$$+ \frac{9}{8} e'^2 f^2 . \cos(3 + 1) + e' e f^2 . \cos(2 + 2) + \frac{9}{8} e^2 f^2 . \cos(1 + 3)$$

Terms of the fifth order,

$$+ \frac{4}{3} e'^3 f^2 . \cos(4 + 1) + \frac{9}{8} e'^2 e f^2 . \cos(3 + 2) + \frac{9}{8} e' e^2 f^2 . \cos(2 + 3)$$

$$+ \frac{4}{3} e^3 f^2 . \cos(1 + 4)$$

SECTION 7.

Development of $\cos(kv' - kv) \cdot f^2 \cdot \cos(v' + v - 2\theta)$, *to the fifth order.*

23. We must multiply the expression in (16), (of which only the terms to the third order will be wanted), by the expression just formed, according to the rule of (8). Thus we obtain the following expression:

Term of the second order,

$$\frac{1}{2} f^2 \cdot \cos(\overline{k+1} - \overline{k-1}).$$

Terms of the third order,

$$\left(\frac{1}{2} k + \frac{1}{2}\right) e' f^2 \cdot \cos(\overline{k+2} - \overline{k-1}) + \left(-\frac{1}{2} k + \frac{1}{2}\right) e f^2 \cos(\overline{k+1} - \overline{k-2}).$$

Terms of the fourth order,

$$\left(\frac{1}{4} k^2 + \frac{13}{16} k + \frac{9}{16}\right) e'^2 f^2 \cdot \cos(\overline{k+3} - \overline{k-1})$$

$$+ \left(-\frac{1}{2} k^2 + \frac{1}{2}\right) e' e f^2 \cdot \cos(\overline{k+2} - \overline{k-2})$$

$$+ \left(\frac{1}{4} k^2 - \frac{13}{16} k + \frac{9}{16}\right) e^2 f^2 \cdot \cos(\overline{k+1} - \overline{k-3})$$

Terms of the fifth order,

$$\left(\frac{1}{12} k^3 + \frac{9}{16} k^2 + \frac{55}{48} k + \frac{2}{3}\right) e'^3 f^2 \cdot \cos(\overline{k+4} - \overline{k-1})$$

$$+ \left(-\frac{1}{4} k^3 - \frac{9}{16} k^2 + \frac{1}{4} k + \frac{9}{16}\right) e'^2 e f^2 \cdot \cos(\overline{k+3} - \overline{k-2})$$

$$+ \left(\frac{1}{4} k^3 - \frac{9}{16} k^2 - \frac{1}{4} k + \frac{9}{16}\right) e' e^2 f^2 \cdot \cos(\overline{k+2} - \overline{k-3})$$

$$+ \left(-\frac{1}{12} k^3 + \frac{9}{16} k^2 - \frac{55}{48} k + \frac{2}{3}\right) e^3 f^2 \cdot \cos(\overline{k+1} - \overline{k-4})$$

SECTION 8.

Selection of the coefficients of cos (13 — 8) *in the development of*

$$-m \cdot \frac{r' r}{\{r'^2 - 2r' r \cdot \cos(v' - v) + r^2\}^{\frac{3}{2}}} f^2 \cdot \cos(v' + v - 2\theta).$$

24. The general term of the expansion is $-m \cdot \Gamma_{\frac{3}{2}}^{(k)} \cdot \cos(kv' - kv) \cdot f^2 \cdot \cos(v' + v - 2\theta)$. The expression for $\cos(kv' - kv) \cdot f^2 \cdot \cos(v' + v - 2\theta)$ we have just found; and the expression for $-\Gamma_{\frac{3}{2}}^{(k)}$ will be in all respects similar to that for $-\Gamma_{\frac{1}{2}}^{(k)}$ in (18), putting $C_{\frac{3}{2}}^{(k)}$ for $C_{\frac{1}{2}}^{(k)}$. Observing that k cannot be less than 9 or greater than 12, and selecting for the different values of k the terms whose combination produces (13 — 8), we get the following coefficients:

$k = 9.$

$$m \times \left\{-\frac{2815}{24}(0,0) + \frac{493}{32}(1,0) - \frac{11}{16}(2,0) + \frac{1}{96}(3,0)\right\} C_{\frac{3}{2}}^{(9)} \cdot e'^3 f^2 \,.\,.\, (M^{(9)} e'^3 f^2)$$

$k = 10.$

$$m \times \left\{\frac{4851}{16}(0,0) - \frac{207}{8}(1,0) + \frac{539}{32}(0,1) + \frac{9}{16}(2,0) - \frac{23}{16}(1,1)\right.$$
$$\left. + \frac{1}{32}(2,1)\right\} C_{\frac{3}{2}}^{(10)} \cdot e'^2 e f^2 \quad \ldots\ldots\ldots\ldots \quad (M^{(10)} \cdot e'^2 e f^2)$$

$k = 11.$

$$m \times \left\{-\frac{525}{2}(0,0) + \frac{175}{16}(1,0) - \frac{57}{2}(0,1) + \frac{19}{16}(1,1) - \frac{3}{4}(0,2)\right.$$
$$\left. + \frac{1}{32}(1,2)\right\} C_{\frac{3}{2}}^{(11)} \cdot e' e^2 f^2 \quad \ldots\ldots\ldots\ldots \quad (M^{(11)} \cdot e' e^2 f^2)$$

$k = 12.$

$$m \times \left\{\frac{913}{12}(0,0) + \frac{97}{8}(0,1) + \frac{5}{8}(0,2) + \frac{1}{96}(0,3)\right\} C_{\frac{3}{2}}^{(12)} \cdot e^3 f^2 \,.\,.\, (M^{(12)} \cdot e^3 f^2)$$

The arguments of the cosines multiplied by these coefficients are not similar; their forms may be found by the reasoning in (10).

25. The next term of R to be developed, by (14), is

$$-m \cdot \frac{3}{4} \frac{r'^2 r^2}{\{r'^2 - 2r'r \cdot \cos(v'-v) + r^2\}^{\frac{5}{2}}} f^4 \cdot \cos(2v' + 2v - 4\theta).$$

We shall put $\Gamma_{\frac{5}{2}}^{(k)}$ for the general term in the expansion

$$\frac{r'^2 r^2}{\{r'^2 - 2r'r \cdot \cos(v'-v) + r^2\}^{\frac{5}{2}}} = \frac{1}{2}\Gamma_{\frac{5}{2}}^{(0)} + \Gamma_{\frac{5}{2}}^{(1)} \cdot \cos(v'-v) + \Gamma_{\frac{5}{2}}^{(2)} \cdot \cos(2v'-2v) + \&c.$$

and $C_{\frac{5}{2}}^{(k)}$ for the general term in the expansion

$$\frac{a'^2 a^2}{\{a'^2 - 2a'a \cdot \cos(v'-v) + a^2\}^{\frac{5}{2}}} = \frac{1}{2}C_{\frac{5}{2}}^{(0)} + C_{\frac{5}{2}}^{(1)} \cdot \cos(v'-v) + C_{\frac{5}{2}}^{(2)} \cdot \cos(2v'-2v) + \&c.$$

SECTION 9.

Development of $\cos(kv'-kv) \cdot f^4 \cdot \cos(2v'+2v-4\theta)$, *to the fifth order.*

26. As the multiplier f^4 is of the fourth order, we need to develope $\cos(2v'+2v-4\theta)$ only to the first order. Now by (13), $2v'+2v-4\theta =$

$$(2+2) - 4\theta$$
$$+ 4e' \cdot \sin(1+0) + 4e \cdot \sin(0+1)$$

and consequently $\cos(2v'+2v-4\theta) =$

$$\cos(2+2-4\theta) - \sin(2+2-4\theta) \cdot \{4e' \cdot \sin(1+0) + 4e \cdot \sin(0+1)\}$$
$$= \cos(2+2-4\theta)$$
$$+ 2e' \cos(3+2-4\theta) + 2e \cdot \cos(2+3-4\theta)$$

Multiplying this by f^4 it will be seen, as in (22), that we may omit 4θ in the argument. Thus we have for the development of $f^4 \cdot \cos(2v'+2v-4\theta)$,

Term of the fourth order,

$$f^4 \cdot \cos(2+2).$$

Terms of the fifth order,

$$+ 2\, e' f^4 . \cos(3 + 2) + 2\, e f^4 . \cos(2 + 3).$$

27. This is now to be multiplied by $\cos(k v' - k v)$, the expansion of which has been performed in (16). Effecting this operation, we have for the development of $\cos(k v' - k v) . f^4 . \cos(2 v' + 2 v - 4 \theta)$,

Term of the fourth order,

$$\frac{1}{2} f^4 . \cos(\overline{k + 2} - \overline{k - 2})$$

Terms of the fifth order,

$$\left(\frac{1}{2} k + 1\right) e' f^4 . \cos(\overline{k + 3} - \overline{k - 2}) + \left(-\frac{1}{2} k + 1\right) e f^4 . \cos(\overline{k + 2} - \overline{k - 3})$$

SECTION 10.

Selection of the coefficients of cos (13 − 8) *in the development of*

$$- m . \frac{3}{4} . \frac{r'^2 r^2}{\{r'^2 - 2 r' r . \cos(v' - v) + r^2\}^{\frac{5}{2}}} . f^4 . \cos(2 v' + 2 v - 4 \theta).$$

28. We must suppose the expression of (27) to be multiplied by $\frac{3}{4} m$, and by the expression for $- \Gamma_{\frac{5}{2}}^{(k)}$ (which will be formed from that of (18), putting $C_{\frac{5}{2}}^{(k)}$ for $C_{\frac{3}{2}}^{(k)}$). Then giving to k different values, we must select the terms in the product whose argument is (13 − 8). It is easily seen that 10 and 11 are the only admissible values of k. Thus we get these coefficients;

$$k = 10.$$

$$m \times \left\{ - \frac{9}{2} (0,0) + \frac{3}{16} (1,0) \right\} C_{\frac{5}{2}}^{(10)} . e' f^4 \quad \ldots\ldots\ldots \quad (N^{(10)} . e' f^4)$$

$$k = 11.$$

$$m \times \left\{ \frac{27}{8} (0,0) + \frac{3}{16} (0,1) \right\} C_{\frac{5}{2}}^{(11)} . e f^4 \quad \ldots\ldots\ldots \quad (N^{(11)} . e f^4)$$

29. The terms collected in (20), (24), and (28), form the complete coefficient of cos (13 − 8) in the development of R to the fifth order. The arguments of the cosines multiplied by the different series are all different; so that there are twelve different terms to be calculated. Using the symbols $\overset{(8)}{\mathrm{L}}$, &c., the complete term is expressed thus:

$$\overset{(8)}{\mathrm{L}} . e'^5 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 5\varpi'\}$$
$$+ \overset{(9)}{\mathrm{L}} . e'^4 e . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 4\varpi' - \varpi\}$$
$$+ \overset{(10)}{\mathrm{L}} . e'^3 e^2 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 3\varpi' - 2\varpi\}$$
$$+ \overset{(11)}{\mathrm{L}} . e'^2 e^3 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 2\varpi' - 3\varpi\}$$
$$+ \overset{(12)}{\mathrm{L}} . e' e^4 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - \varpi' - 4\varpi\}$$
$$+ \overset{(13)}{\mathrm{L}} . e^5 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 5\varpi\}$$
$$+ \overset{(9)}{\mathrm{M}} . e'^3 f^2 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 3\varpi' - 2\theta\}$$
$$+ \overset{(10)}{\mathrm{M}} . e'^2 e f^2 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 2\varpi' - \varpi - 2\theta\}$$
$$+ \overset{(11)}{\mathrm{M}} . e' e^2 f^2 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - \varpi' - 2\varpi - 2\theta\}$$
$$+ \overset{(12)}{\mathrm{M}} . e^3 f^2 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - 3\varpi - 2\theta\}$$
$$+ \overset{(10)}{\mathrm{N}} . e' f^4 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - \varpi' - 4\theta\}$$
$$+ \overset{(11)}{\mathrm{N}} . e f^4 . \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon) - \varpi - 4\theta\}$$

SECTION 11.

Considerations on the numerical calculation of the inequalities in the Earth's motion depending on this term.

30. If we examine the expressions of (2), it will appear that the values of all may be deduced with little trouble from the terms above, except that depending on $\frac{d\mathrm{R}}{da'}$. Since a' enters only into the coefficients, $\frac{d\mathrm{R}}{da'}$ will be produced by

differentiating the coefficients and retaining the same cosines. The coefficients will be differentiated by changing $(0,0)\ C_{\frac{1}{2}}^{(8)}$ into $\frac{1}{a'}(1,0)\ C_{\frac{1}{2}}^{(8)}$, $(3,2)\ C_{\frac{1}{2}}^{(8)}$ into $\frac{1}{a'}(4,2)\ C_{\frac{1}{2}}^{(8)} + \frac{3}{a'}(3,2)\ C_{\frac{1}{2}}^{(8)}$, &c. Thus new terms will be introduced whose calculation is rather troublesome. It is desirable, then, to inquire whether it is probable that the term depending on $\frac{d\,R}{d\,a'}$ will be comparable in magnitude to the other term which has the same argument.

31. Now if we put $A \,.\, \cos\{13\,(n'\,t + \varepsilon') - 8\,(n\,t + \varepsilon) + B\}$ or $A\cos(13 - 8)$, for one of the terms, we find

$$\frac{d\,n'}{d\,t} = -\,3\,.\,13\,.\,\frac{n'^2\,a'}{\mu'}\,.\,A\,.\,\sin(13 - 8)$$

whence

$$n' = N' + \frac{3\,.\,13\,.\,n'^2\,a'}{(13\,n' - 8\,n)\,\mu'}\,A\,.\,\cos(13 - 8)$$

(where N' is constant and $=$ mean value of n')

$$\frac{d\,\varepsilon'}{d\,t} = +\,3\,.\,13\,.\,\frac{n'^2\,a'}{\mu'}\,A\,.\,t\,.\sin(13 - 8) + \frac{2\,n'\,a'^2}{\mu'}\,.\,\frac{d\,A}{d\,a'}\cos(13 - 8)$$

whence

$$\varepsilon' = E' - \frac{3\,.\,13\,n'^2\,a'}{(13\,n' - 8\,n)\,\mu'}\,A\,.\,t\,.\cos(13 - 8) + \frac{3\,.\,13\,n'^2\,a'}{(13\,n' - 8\,n)^2\,\mu'}\,A\,.\sin(13 - 8)$$
$$+ \frac{2\,n'\,a'^2}{(13\,n' - 8\,n)\,\mu'}\,.\,\frac{d\,A}{d\,a'}\sin(13 - 8)$$

(where E' is constant and $=$ mean value of ε')

and $n'\,t + \varepsilon'$ (which, by (1), is the first term of v') becomes

$$N'\,t + E' + \left\{\frac{3\,.\,13\,n'^2\,a'}{(13\,n' - 8\,n)^2\,\mu'}\,A + \frac{2\,n'\,a'^2}{(13\,n' - 8\,n)\,\mu'}\,.\,\frac{d\,A}{d\,a'}\right\}\sin(13 - 8).$$

The ratio of the two coefficients of the inequality $\sin(13 - 8)$ is

$$\frac{39}{2}\,.\,\frac{n'}{13\,n' - 8\,n}\,A : a'\,\frac{d\,A}{d\,a'}$$

or nearly $\qquad 4800 \times A : a'\,\frac{d\,A}{d\,a'}.$

It will be seen hereafter, that for any one of the terms whose union composes $\overset{(8)}{L}$, &c., $a'\frac{dA}{da'}$ is greater than $-$ A, and that it may, on the mean of values, be said to differ little from -12 A. This reduces the ratio of the terms to 400 : 1. Now though we cannot assert that the sum of one set of terms will have to the sum of the other set of terms a ratio at all similar to this, yet the great disproportion of the terms related to each other seems sufficiently to justify us in the *à priori* assertion that the terms depending on $\frac{dR}{da'}$ are not worth calculating. It will readily be seen that the terms depending on $\frac{dR}{de'}$ are still more insignificant than those depending on $\frac{dR}{da'}$.

32. We stated in (1) that the variations of the elements would be sufficiently taken into account in the expression for R if we put E + F t for e, &c.; which amounts to taking only the secular variations. There will be no difficulty in doing this for e', e, ϖ', ϖ, f, and θ: but if such terms existed in the approximate expressions for a' and a, they would require the use of the differentials $\frac{dR}{da}$, $\frac{dR}{da}$. But a' and a have no secular variations: and therefore these differentials are not wanted. We may therefore proceed at once with the numerical calculation of the terms $\overset{(8)}{L}$, $\overset{(9)}{L}$, &c.

Section 12.

Numerical calculation of $\overset{(0)}{C}_{\frac{1}{2}}$, $\overset{(1)}{C}_{\frac{1}{2}}$, $\overset{(2)}{C}_{\frac{1}{2}}$, *&c.*, $\overset{(k)}{C}_{\frac{3}{2}}$, $\overset{(k)}{C}_{\frac{5}{2}}$, *&c. to* $\overset{(k)}{C}_{\frac{11}{2}}$.

33. If we put $\pi - 2\omega$ for $v' - v$, we have

$$\frac{1}{\sqrt{\{a'^2 + 2a'a \,.\, \cos 2\omega + a^2\}}} = \tfrac{1}{2}\overset{(0)}{C}_{\frac{1}{2}} - \overset{(1)}{C}_{\frac{1}{2}} \,.\, \cos 2\omega + \overset{(2)}{C}_{\frac{1}{2}} \,.\, \cos 4\omega - \text{\&c.}$$

Integrating both sides with respect to ω, from $\omega = 0$ to $\omega = \frac{\pi}{2}$, and putting S_ω for the symbol of integration with respect to ω between these limits,

$$S_\omega \,.\, \frac{1}{\sqrt{\{a'^2 + 2a'a \,.\, \cos 2\omega + a^2\}}} = \frac{\pi}{4}\overset{(0)}{C}_{\frac{1}{2}}$$

whence

$$C_{\frac{1}{2}}^{(0)} = \frac{4}{\pi} \, S_{\omega} \cdot \frac{1}{\sqrt{\{a'^2 + 2\,a'\,a \,.\, \cos 2\,\omega + a^2\}}}$$

or, putting α for $\frac{a}{a'}$,

$$C_{\frac{1}{2}}^{(0)} = \frac{4}{\pi\,a'} \cdot S_{\omega} \frac{1}{\sqrt{\{1 + 2\,\alpha\cos 2\,\omega + \alpha^2\}}}.$$

Now let $\sin\omega' = \frac{\sin 2\,\omega}{\sqrt{\{1 + 2\,\alpha\cos 2\,\omega + \alpha^2\}}}$; and $\alpha' = \frac{1 - \sqrt{1 - \alpha^2}}{1 + \sqrt{1 - \alpha^2}}$: after substitution it is found that

$$C_{\frac{1}{2}}^{(0)} = \frac{4}{\pi\,a'}(1 + \alpha') \,.\, S_{\omega'} \cdot \frac{1}{\sqrt{\{1 + 2\,\alpha'\cos 2\,\omega' + \alpha'^2\}}}$$

In the same manner, making $\sin\omega'' = \frac{\sin 2\,\omega'}{\sqrt{\{1 + 2\,\alpha'\cos 2\,\omega' + \alpha'^2\}}}$: $\alpha'' = \frac{1 - \sqrt{1 - \alpha'^2}}{1 + \sqrt{1 - \alpha'^2}}$: and so on, we get for $C_{\frac{1}{2}}^{(0)}$ the expression

$$\frac{4}{\pi\,a'}(1 + \alpha')\,(1 + \alpha'') \ldots\ldots (1 + \alpha^{(n)}) \,.\, S_{\omega^{(n)}} \frac{1}{\sqrt{\{1 + 2\,\alpha^{(n)}\cos 2\,\omega^{(n)} + \alpha^{(n)2}\}}}$$

The values of α', α'', &c. decrease very rapidly; and when $\alpha^{(n)}$ is insensible, $S_{\omega^{(n)}} \frac{1}{\sqrt{\{1 + 2\,\alpha^{(n)}\cos 2\,\omega^{(n)} + \alpha^{(n)2}\}}}$ becomes $S_{\omega^{(n)}} \,.\, 1$ or $\frac{\pi}{2}$. Consequently

$$C_{\frac{1}{2}}^{(0)} = \frac{2}{a'}\,(1 + \alpha')\,(1 + \alpha'')\,(1 + \alpha''') \,.\, \&c.$$

the factors being continued till $\alpha^{(n)}$ becomes insensible. The calculation is very easy; for, if we make $\sin\beta = \alpha$, $\sin\beta' = \tan^2\frac{\beta}{2}$, $\sin\beta'' = \tan^2\frac{\beta'}{2}$, &c. then $C_{\frac{1}{2}}^{(0)}$ $= \frac{2}{a'}\sec^2\frac{\beta}{2} \,.\, \sec^2\frac{\beta'}{2} \,.\, \sec^2\frac{\beta''}{2} \,.\, \&c.$ For Venus and the Earth (Méc. Cél. liv. VI.) α or $\frac{a}{a'} = 0{,}7233323$: using this number in the calculation, $C_{\frac{1}{2}}^{(0)} = \frac{1}{a'} \times 2{,}386375$.

34. Again, $\frac{\cos 2\,\omega}{\sqrt{\{a'^2 + 2\,a'\,a \,.\, \cos 2\,\omega + a^2\}}} = \frac{1}{2}\,C_{\frac{1}{2}}^{(0)} \,.\, \cos 2\,\omega - \frac{1}{2}\,C_{\frac{1}{2}}^{(1)}(1 + \cos 4\,\omega)$ $+ \frac{1}{2}\,C_{\frac{1}{2}}^{(2)}(\cos 2\,\omega + \cos 6\,\omega) - \&c.$; integrating between the same limits as before,

$$C_{\frac{1}{2}}^{(1)} = -\frac{4}{\pi a'} S_\omega \cdot \frac{\cos 2\omega}{\sqrt{\{1 + 2\alpha\cos 2\omega + \alpha^2\}}}$$

Making the same substitution as in (33), there will be produced three terms; of which one vanishes in the definite integral, the second is similar to the expression of this article, and the third similar to that of (33). Making a similar substitution in the second term, new terms are produced. Pursuing this method, it will be found that the only terms whose values are ultimately sensible are those which are similar to the expression of (33): and at last we get

$$C_{\frac{1}{2}}^{(1)} = C_{\frac{1}{2}}^{(0)} \cdot \left\{\frac{\sin\beta}{2} + \frac{\sin\beta}{2}\cdot\frac{\sin\beta'}{2} + \frac{\sin\beta}{2}\cdot\frac{\sin\beta'}{2}\cdot\frac{\sin\beta''}{2} + \&c.\right\} = \frac{1}{a'} \times 0{,}9424137$$

35. Putting χ for $v'-v$, and differentiating with respect to χ the logarithms of both sides of the equation

$$\frac{1}{\sqrt{\{a'^2 - 2a'a\cdot\cos\chi + a^2\}}} = \frac{1}{2} C_{\frac{1}{2}}^{(0)} + C_{\frac{1}{2}}^{(1)}\cdot\cos\chi + C_{\frac{1}{2}}^{(2)}\cos 2\chi + \&c.$$

multiplying out the denominators, and comparing the coefficients of $\cos k\chi$,

$$C_{\frac{1}{2}}^{(k+1)} = \frac{2k}{2k+1}\left(\frac{1}{\alpha} + \alpha\right) C_{\frac{1}{2}}^{(k)} - \frac{2k-1}{2k+1} C_{\frac{1}{2}}^{(k-1)}$$

where $\frac{1}{\alpha} + \alpha = 2{,}1058226$. Making k successively 1, 2, 3, 4, &c., we get the following values:

$C_{\frac{1}{2}}^{(0)} = \frac{1}{a'} \times 2{,}3863750$* $\quad C_{\frac{1}{2}}^{(6)} = \frac{1}{a'} \times 0{,}0903724 \quad C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 0{,}0093812$

$C_{\frac{1}{2}}^{(1)} = \frac{1}{a'} \times 0{,}9424137 \quad C_{\frac{1}{2}}^{(7)} = \frac{1}{a'} \times 0{,}0609432 \quad C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 0{,}0065274$

$C_{\frac{1}{2}}^{(2)} = \frac{1}{a'} \times 0{,}5275791 \quad C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 0{,}0414571 \quad C_{\frac{1}{2}}^{(14)} = \frac{1}{a'} \times 0{,}0045503$

$C_{\frac{1}{2}}^{(3)} = \frac{1}{a'} \times 0{,}3233422 \quad C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 0{,}0283925 \quad C_{\frac{1}{2}}^{(15)} = \frac{1}{a'} \times 0{,}0031744$

$C_{\frac{1}{2}}^{(4)} = \frac{1}{a'} \times 0{,}2067875 \quad C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 0{,}0195495 \quad C_{\frac{1}{2}}^{(16)} = \frac{1}{a'} \times 0{,}0022123$

$C_{\frac{1}{2}}^{(5)} = \frac{1}{a'} \times 0{,}1355852 \quad C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 0{,}0135189 \quad C_{\frac{1}{2}}^{(17)} = \frac{1}{a'} \times 0{,}0015356$

$C_{\frac{1}{2}}^{(18)} = \frac{1}{a'} \times 0{,}0010554$

* LAPLACE's numbers, which are somewhat different from these, are computed by the less accurate method of summing a slowly converging series.

36. For the calculation of the terms $C_{\frac{3}{2}}^{(k)}$, $C_{\frac{5}{2}}^{(k)}$, &c., we shall adopt the general notation

$$\frac{1}{\sqrt{a'a}\,.\left\{\frac{a'}{a}-2\cos\chi+\frac{a}{a'}\right\}^{s}}=\frac{1}{2}\,C_s^{(0)}+C_s^{(1)}\cos\chi+C_s^{(2)}\cos 2\chi+\&c.$$

which, it will be seen, includes those of (17), (21), and (25); and proceeding as in (35) we shall find this general equation

$$C_s^{(k+1)}=\frac{k}{k+1-s}\left(\frac{1}{\alpha}+\alpha\right)C_s^{(k)}-\frac{k-1+s}{k+1-s}\,C_s^{(k-1)}$$

And since $\dfrac{1}{\sqrt{a'a}\,.\left\{\frac{a'}{a}-2\cos\chi+\frac{a}{a'}\right\}^{s}}=$

$$\left(\frac{1}{\alpha}+\alpha-2\cos\chi\right)\times\frac{1}{\sqrt{a'a}\,.\left\{\frac{a'}{a}-2\cos\chi+\frac{a}{a'}\right\}^{s+1}}$$

we find on substituting the expansions and comparing the coefficients of $\cos k\chi$,

$$C_s^{(k)}=\left(\frac{1}{\alpha}+\alpha\right)C_{s+1}^{(k)}-C_{s+1}^{(k-1)}-C_{s+1}^{(k+1)}$$

Removing $C_{s+1}^{(k+1)}$ by means of the relation just found (putting $s+1$ for s)

$$C_s^{(k)}=-\frac{s}{k-s}\left(\frac{1}{\alpha}+\alpha\right)C_{s+1}^{(k)}+\frac{2s}{k-s}\,C_{s+1}^{(k-1)}$$

In nearly the same manner,

$$C_s^{(k-1)}=\frac{s}{k+s-1}\left(\frac{1}{\alpha}+\alpha\right)C_{s+1}^{(k-1)}-\frac{2s}{k+s-1}\,C_{s+1}^{(k)}$$

Eliminating $C_{s+1}^{(k-1)}$,

$$C_{s+1}^{(k)}=\frac{2(k+s-1)}{s}\cdot\frac{1}{\left(\frac{1}{\alpha}-\alpha\right)^2}\,C_s^{(k-1)}-\frac{k-s}{s}\cdot\frac{\frac{1}{\alpha}+\alpha}{\left(\frac{1}{\alpha}-\alpha\right)^2}\,C_s^{(k)}$$

If in this we substitute the value of $C_s^{(k)}$ in terms of $C_s^{(k-1)}$ and $C_s^{(k+1)}$, given by the relation above,

$$C^{(k)}_{s+1} = \frac{1}{\left(\frac{1}{\alpha} - \alpha\right)^2} \left\{ \frac{(k+s)(k+s-1)}{ks} C^{(k-1)}_s - \frac{(k-s)(k-s+1)}{ks} C^{(k+1)}_s \right\}$$

in which $\frac{1}{\left(\frac{1}{\alpha} - \alpha\right)^2}$ or $\frac{\alpha^2}{(1-\alpha^2)^2}$ is $\frac{\sin^2\beta}{\cos^4\beta} = 2{,}3015505$

37. Making $s = \frac{1}{2}$, $C^{(k)}_{\frac{3}{2}} = \frac{\sin^2\beta}{2\cos^4\beta} \cdot \frac{4k^2-1}{k} \cdot \left(C^{(k-1)}_{\frac{1}{2}} - C^{(k+1)}_{\frac{1}{2}}\right)$. Using this formula,

$C^{(4)}_{\frac{3}{2}} = \frac{1}{a'} \times 3{,}403041$ $\quad C^{(9)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}904785$ $\quad C^{(14)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}215803$

$C^{(5)}_{\frac{3}{2}} = \frac{1}{a'} \times 2{,}652559$ $\quad C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}682935$ $\quad C^{(15)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}161251$

$C^{(6)}_{\frac{3}{2}} = \frac{1}{a'} \times 2{,}047192$ $\quad C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}513799$ $\quad C^{(16)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}120579$

$C^{(7)}_{\frac{3}{2}} = \frac{1}{a'} \times 1{,}568093$ $\quad C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}385521$ $\quad C^{(17)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}090452$

$C^{(8)}_{\frac{3}{2}} = \frac{1}{a'} \times 1{,}193991$ $\quad C^{(13)}_{\frac{3}{2}} = \frac{1}{a'} \times 0{,}288655$

38. Making $s = \frac{3}{2}$, $C^{(k)}_{\frac{5}{2}} = \frac{\sin^2\beta}{6\cos^4\beta} \cdot \left\{ \frac{(2k+3)(2k+1)}{k} C^{(k-1)}_{\frac{3}{2}} - \frac{(2k-3)(2k-1)}{k} C^{(k+1)}_{\frac{3}{2}} \right\}$. By the use of this formula,

$C^{(5)}_{\frac{5}{2}} = \frac{1}{a'} \times 27{,}43922$ $\quad C^{(9)}_{\frac{5}{2}} = \frac{1}{a'} \times 12{,}88246$ $\quad C^{(13)}_{\frac{5}{2}} = \frac{1}{a'} \times 5{,}24565$

$C^{(6)}_{\frac{5}{2}} = \frac{1}{a'} \times 23{,}14387$ $\quad C^{(10)}_{\frac{5}{2}} = \frac{1}{a'} \times 10{,}39741$ $\quad C^{(14)}_{\frac{5}{2}} = \frac{1}{a'} \times 4{,}12790$

$C^{(7)}_{\frac{5}{2}} = \frac{1}{a'} \times 19{,}25046$ $\quad C^{(11)}_{\frac{5}{2}} = \frac{1}{a'} \times 8{,}32969$ $\quad C^{(15)}_{\frac{5}{2}} = \frac{1}{a'} \times 3{,}23120$

$C^{(8)}_{\frac{5}{2}} = \frac{1}{a'} \times 15{,}82608$ $\quad C^{(12)}_{\frac{5}{2}} = \frac{1}{a'} \times 6{,}62955$ $\quad C^{(16)}_{\frac{5}{2}} = \frac{1}{a'} \times 2{,}51561$

39. Making $s = \frac{5}{2}$, $C^{(k)}_{\frac{7}{2}} = \frac{\sin^2\beta}{10\cos^2\beta} \cdot \left\{ \frac{(2k+5)(2k+3)}{k} C^{(k-1)}_{\frac{5}{2}} - \frac{(2k-5)(2k-3)}{k} C^{(k+1)}_{\frac{5}{2}} \right\}$. Thus we get

$C_{\frac{7}{2}}^{(6)} = \frac{1}{a'} \times 221{,}8780$ $C_{\frac{7}{2}}^{(9)} = \frac{1}{a'} \times 143{,}6296$ $C_{\frac{7}{2}}^{(12)} = \frac{1}{a'} \times 84{,}9489$

$C_{\frac{7}{2}}^{(7)} = \frac{1}{a'} \times 194{,}2735$ $C_{\frac{7}{2}}^{(10)} = \frac{1}{a'} \times 121{,}5988$ $C_{\frac{7}{2}}^{(13)} = \frac{1}{a'} \times 70{,}2184$

$C_{\frac{7}{2}}^{(8)} = \frac{1}{a'} \times 167{,}9770$ $C_{\frac{7}{2}}^{(11)} = \frac{1}{a'} \times 102{,}0404$ $C_{\frac{7}{2}}^{(14)} = \frac{1}{a'} \times 57{,}6762$

$C_{\frac{7}{2}}^{(15)} = \frac{1}{a'} \times 47{,}1003$

40. Making $s = \frac{7}{2}$, $C_{\frac{9}{2}}^{(k)} = \frac{\sin^2\beta}{14\cos^2\beta}\left\{\frac{(2k+7)(2k+5)}{k} C_{\frac{7}{2}}^{(k-1)} - \frac{(2k-7)(2k-5)}{k} C_{\frac{7}{2}}^{(k+1)}\right\}$. From this,

$C_{\frac{9}{2}}^{(7)} = \frac{1}{a'} \times 1830{,}596$ $C_{\frac{9}{2}}^{(10)} = \frac{1}{a'} \times 1266{,}709$ $C_{\frac{9}{2}}^{(13)} = \frac{1}{a'} \times 807{,}945$

$C_{\frac{9}{2}}^{(8)} = \frac{1}{a'} \times 1636{,}049$ $C_{\frac{9}{2}}^{(11)} = \frac{1}{a'} \times 1099{,}213$ $C_{\frac{9}{2}}^{(14)} = \frac{1}{a'} \times 685{,}214$

$C_{\frac{9}{2}}^{(9)} = \frac{1}{a'} \times 1446{,}655$ $C_{\frac{9}{2}}^{(12)} = \frac{1}{a'} \times 946{,}016$

41. Making $s = \frac{9}{2}$, $C_{\frac{11}{2}}^{(k)} = \frac{\sin^2\beta}{18\cos^2\beta}\left\{\frac{(2k+9)(2k+7)}{k} C_{\frac{9}{2}}^{(k-1)} - \frac{(2k-9)(2k-7)}{k} C_{\frac{9}{2}}^{(k+1)}\right\}$. From this,

$C_{\frac{11}{2}}^{(8)} = \frac{1}{a'} \times 15366{,}90$ $C_{\frac{11}{2}}^{(10)} = \frac{1}{a'} \times 12473{,}68$ $C_{\frac{11}{2}}^{(12)} = \frac{1}{a'} \times 9786{,}59$

$C_{\frac{11}{2}}^{(9)} = \frac{1}{a'} \times 13907{,}74$ $C_{\frac{11}{2}}^{(11)} = \frac{1}{a'} \times 11092{,}76$ $C_{\frac{11}{2}}^{(13)} = \frac{1}{a'} \times 8570{,}07$

SECTION 13.

Numerical calculation of $(0,1)\ C_s^{(k)}$, $(1,0)\ C_s^{(k)}$, *&c.*

42. It will be sufficient to form, by differentiation, the expression for one of the differential coefficients of each order, as the others can then be derived by simple addition. For $C_s^{(k)}$ is a function of a' and a of -1 dimension: hence

$a' \frac{d}{d a'} C_s^{(k)} + a \frac{d}{d a} C_s^{(k)} = - C_s^{(k)}$, or $(1,0)\, C_s^{(k)} + (0,1)\, C_s^{(k)} = - C_s^{(k)}$. Again (as another instance) $\frac{d^4}{d a'^3 . d a} C_s^{(k)}$ is a function of a' and a of -5 dimensions; consequently $a' \frac{d^5}{d a'^4 . d a} C_s^{(k)} + a \frac{d^5}{d a'^3 . d a^2} C_s^{(k)} = -5 \frac{d^4}{d a'^3 . d a} C_s^{(k)}$: or, multiplying both sides by $a'^3 a$, $(4,1)\, C_s^{(k)} + (3,2)\, C_s^{(k)} = -5\, (3,1)\, C_s^{(k)}$. It is indifferent which coefficient of each order we calculate first; and for the algebraical process it is rather most convenient to differentiate successively with regard to the same quantity (as a').

43. Now $\frac{d}{d a'} \left\{ \frac{1}{\sqrt{a' a}} \cdot \frac{1}{\left(\frac{a'}{a} + \frac{a}{a'} - 2 \cos \chi\right)^s} \right\} =$

$$- \frac{1}{2} \cdot \frac{1}{a'} \cdot \frac{1}{\sqrt{a' a}} \cdot \frac{1}{\left(\frac{a'}{a} + \frac{a}{a'} - 2 \cos \chi\right)^s}$$

$$+ s \left(-\frac{1}{a} + \frac{a}{a'^2}\right) \frac{1}{\sqrt{a' a}} \cdot \frac{1}{\left(\frac{a'}{a} + \frac{a}{a'} - 2 \cos \chi\right)^{s+1}}$$

or, taking the coefficient of $\cos k \chi$ in the expansion on both sides,

$$\frac{d}{d a'} C_s^{(k)} = - \frac{1}{2} \cdot \frac{1}{a'} C_s^{(k)} + \left(-\frac{1}{a} + \frac{a}{a'^2}\right) s . C_{s+1}^{(k)}$$

Differentiating this formula with respect to a', and using the same formula to simplify the differential coefficient, we get $\frac{d^2}{d a'^2} C_s^{(k)}$. In the same manner $\frac{d^3}{d a'^3} C_s^{(k)}$, &c. are found; multiplying them $\left(\text{beginning with } \frac{d}{d a'} C_s^{(k)}\right)$ by a', a'^2, a'^3, &c., we obtain the following expressions:

$$(1,0)\, C_s^{(k)} = - \frac{1}{2} C_s^{(k)} + \left(-\frac{1}{\alpha} + \alpha\right) s . C_{s+1}^{(k)}$$

$$(2,0)\, C_s^{(k)} = + \frac{3}{4} C_s^{(k)} + \left(\frac{1}{\alpha} - 3\alpha\right) s . C_{s+1}^{(k)} + \left(-\frac{1}{\alpha} + \alpha\right)^2 . s . \overline{s+1} . C_{s+2}^{(k)}$$

$$(3,0)\, C_s^{(k)} = - \frac{15}{8} C_s^{(k)} + \frac{9}{4} \left(-\frac{1}{\alpha} + 5\alpha\right) s . C_{s+1}^{(k)}$$

$$+\frac{3}{2}\left(\frac{1}{\alpha}-\alpha\right)\left(-\frac{1}{\alpha}+5\alpha\right).s.\overline{s+1}.C^{(k)}_{s+2}$$

$$+\left(-\frac{1}{\alpha}+\alpha\right)^3.s.\overline{s+1}.\overline{s+2}.C^{(k)}_{s+3}$$

$$(4,0)\ C^{(k)}_s=+\frac{105}{16}C^{(k)}_s+\frac{15}{2}\left(\frac{1}{\alpha}-7\alpha\right)s.C^{(k)}_{s+1}$$

$$+\left\{\left(\frac{9}{2}.\frac{1}{\alpha}-\frac{15}{2}\alpha\right).\left(\frac{1}{\alpha}-7\alpha\right)-6\right\}.s.\overline{s+1}.C^{(k)}_{s+2}$$

$$+2\left(-\frac{1}{\alpha}+\alpha\right)^2.\left(\frac{1}{\alpha}-7\alpha\right).s.\overline{s+1}.\overline{s+2}.C^{(k)}_{s+3}$$

$$+\left(-\frac{1}{\alpha}+\alpha\right)^4.s.\overline{s+1}.\overline{s+2}.\overline{s+3}.C^{(k)}_{s+4}$$

$$(5,0)\ C^{(k)}_s=-\frac{945}{32}C^{(k)}_s+\frac{525}{16}\left(-\frac{1}{\alpha}+9\alpha\right)s.C^{(k)}_{s+1}$$

$$+\frac{75}{4}\left\{-\left(\frac{1}{\alpha}-3\alpha\right)\left(\frac{1}{\alpha}-7\alpha\right)+4\right\}s.\overline{s+1}.C^{(k)}_{s+2}$$

$$+\frac{15}{2}\left(\frac{1}{\alpha}-\alpha\right)\left\{-\left(\frac{1}{\alpha}-3\alpha\right)\left(\frac{1}{\alpha}-7\alpha\right)+4\right\}s.\overline{s+1}.\overline{s+2}.C^{(k)}_{s+3}$$

$$+\frac{5}{2}\left(\frac{1}{\alpha}-\alpha\right)^3.\left(-\frac{1}{\alpha}+9\alpha\right)s.\overline{s+1}.\overline{s+2}.\overline{s+3}.C^{(k)}_{s+4}$$

$$+\left(-\frac{1}{\alpha}+\alpha\right)^5.s.\overline{s+1}.\overline{s+2}.\overline{s+3}.\overline{s+4}.C^{(k)}_{s+5}$$

44. Using the same value of α as before, and making $s=\frac{1}{2}$, these expressions become,

$$(1,0)\ C^{(k)}_{\frac{1}{2}}=-\frac{1}{2}C^{(k)}_{\frac{1}{2}}-0{,}3295790.C^{(k)}_{\frac{3}{2}}$$

$$(2,0)\ C^{(k)}_{\frac{1}{2}}=+\frac{3}{4}C^{(k)}_{\frac{1}{2}}-0{,}3937533.C^{(k)}_{\frac{3}{2}}+0{,}3258670.C^{(k)}_{\frac{5}{2}}$$

$$(3,0)\ C^{(k)}_{\frac{1}{2}}=-\frac{15}{8}C^{(k)}_{\frac{1}{2}}+2{,}5134426.C^{(k)}_{\frac{3}{2}}+1{,}6567557.C^{(k)}_{\frac{5}{2}}-0{,}5369945.C^{(k)}_{\frac{7}{2}}$$

$$(4,0)\ C_{\frac{1}{2}}^{(k)} = +\frac{105}{16}\,C_{\frac{1}{2}}^{(k)} - 13,8031342\,.\,C_{\frac{3}{2}}^{(k)} - 6,6980500\,.\,C_{\frac{5}{2}}^{(k)}$$

$$- 5,9973139\,.\,C_{\frac{7}{2}}^{(k)} + 1,2388750\,.\,C_{\frac{9}{2}}^{(k)}$$

$$(5,0)\ C_{\frac{1}{2}}^{(k)} = -\frac{945}{32}\,C_{\frac{1}{2}}^{(k)} + 84,1230534\,.\,C_{\frac{3}{2}}^{(k)} + 15,4872787\,.\,C_{\frac{5}{2}}^{(k)}$$

$$+ 10,2085636\,.\,C_{\frac{7}{2}}^{(k)} + 24,0925995\,.\,C_{\frac{9}{2}}^{(k)} - 3,6747654\,.\,C_{\frac{11}{2}}^{(k)}$$

45. Making $s = \frac{3}{2}$, the formulæ give

$$(1,0)\ C_{\frac{3}{2}}^{(k)} = -\frac{1}{2}\,C_{\frac{3}{2}}^{(k)} - 0,9887370\,.\,C_{\frac{5}{2}}^{(k)}$$

$$(2,0)\ C_{\frac{3}{2}}^{(k)} = +\frac{3}{4}\,C_{\frac{3}{2}}^{(k)} - 1,1812599\,.\,C_{\frac{5}{2}}^{(k)} + 1,6293350\,.\,C_{\frac{7}{2}}^{(k)}$$

$$(3,0)\ C_{\frac{3}{2}}^{(k)} = -\frac{15}{8}\,C_{\frac{3}{2}}^{(k)} + 7,5403278\,.\,C_{\frac{5}{2}}^{(k)} + 8,2837785\,.\,C_{\frac{7}{2}}^{(k)} - 3,7589615\,.\,C_{\frac{9}{2}}^{(k)}$$

46. Making $s = \frac{5}{2}$, the first formula gives

$$(1,0)\ C_{\frac{5}{2}}^{(k)} = -\frac{1}{2}\,C_{\frac{5}{2}}^{(k)} - 1,6478950\,.\,C_{\frac{7}{2}}^{(k)}.$$

47. Substituting in these the values of $C_{\frac{1}{2}}^{(k)}$, $C_{\frac{3}{2}}^{(k)}$, &c. found in the last section for different values of k, we form the following tables:

For the development of the first term,

$k = 8$

$$(0,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 0,0414571$$

$$(1,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times -0,414243$$

$$(2,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 4,71815$$

$$(3,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times -61,0595$$

$(4,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times 897,236$

$(5,0)\ C_{\frac{1}{2}}^{(8)} = \frac{1}{a'} \times -14993,97$

$k = 9$ $\quad (0,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 0,0283925$

$(1,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -0,312394$ $\quad (0,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 0,284001$

$(2,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 3,86300$ $\quad (1,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -3,23821$

$(3,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -53,5643$ $\quad (2,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 41,9753$

$(4,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 832,244$ $\quad (3,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -617,987$

$(5,0)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times -14512,93$ $\quad (4,1)\ C_{\frac{1}{2}}^{(9)} = \frac{1}{a'} \times 10351,71$

$k = 10$ $\quad (0,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 0,0195495$

$(1,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -0,234856$ $\quad (0,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 0,215306$

$(2,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 3,13393$ $\quad (1,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -2,66422$ $\quad (0,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 2,23361$

$(3,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -46,3921$ $\quad (2,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 36,9903$ $\quad (1,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -28,9976$

$(4,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 761,088$ $\quad (3,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -575,520$ $\quad (2,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 427,559$

$(5,0)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -13860,27$ $\quad (4,1)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times 10054,83$ $\quad (3,2)\ C_{\frac{1}{2}}^{(10)} = \frac{1}{a'} \times -7177,23$

$k = 11$ $\quad (0,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 0,0135189$

$(1,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -0,176097$ $\quad (0,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 0,162578$

$(2,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 2,52220$ $\quad (1,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -2,17001$ $\quad (0,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 1,84485$

$(3,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -39,7288$ $\quad (2,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 32,1622$ $\quad (1,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -25,6522$

$(0,3)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 20,1176$ $\quad (4,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 687,024$ $\quad (3,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -528,109$

$(2,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 399,460$ $\quad (1,3)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -296,851$ $\quad (5,0)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -13066,87$

$(4,1)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 9631,75$ $(3,2)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times -6991,20$ $(2,3)\ C_{\frac{1}{2}}^{(11)} = \frac{1}{a'} \times 4993,90$

$k=12$ $(0,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 0,0093812$

$(1,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -0,131750$ $(0,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 0,122369$

$(2,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 2,01559$ $(1,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -1,75209$ $(0,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 1,50735$

$(3,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -33,6822$ $(2,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 27,6354$ $(1,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -22,3791$

$(0,3)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 17,8570$ $(4,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 612,866$ $(3,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -478,137$

$(2,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 367,595$ $(1,3)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -278,079$ $(0,4)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 206,651$

$(5,0)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -12169,39$ $(4,1)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 9105,06$ $(3,2)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -6714,37$

$(2,3)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times 4876,40$ $(1,4)\ C_{\frac{1}{2}}^{(12)} = \frac{1}{a'} \times -3486,00$

$k=13$ $(0,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 0,0065274$

$(1,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -0,098398$ $(0,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 0,091871$

$(2,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 1,60062$ $(1,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -1,40382$ $(0,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 1,22008$

$(3,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -28,3028$ $(2,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 23,5009$ $(1,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -19,2894$

$(0,3)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 15,6292$ $(4,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 540,744$ $(3,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -427,533$

$(2,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 333,529$ $(1,3)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -256,371$ $(0,4)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 193,854$

$(5,0)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -11205,35$ $(4,1)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 8501,63$ $(3,2)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -6363,96$

$(2,3)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 4696,32$ $(1,4)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times -3414,46$ $(0,5)\ C_{\frac{1}{2}}^{(13)} = \frac{1}{a'} \times 2445,19$

For the development of the second term,

$k=9$ $(0,0)\ C_{\frac{3}{2}}^{(9)} = \frac{1}{a'} \times 0,904785$

$(1,0)\ C_{\frac{3}{2}}^{(9)} = \frac{1}{a'} \times -13,18976$

$$(2,0)\ C^{(9)}_{\frac{3}{2}} = \frac{1}{a'} \times 219,4819$$

$$(3,0)\ C^{(9)}_{\frac{3}{2}} = \frac{1}{a'} \times -4152,686$$

$k = 10$

$$(0,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 0,682935$$

$$(1,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times -10,62177 \qquad (0,1)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 9,93883$$

$$(2,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 186,3554 \qquad (1,1)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times -165,1119$$

$$(3,0)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times -3677,095 \qquad (2,1)\ C^{(10)}_{\frac{3}{2}} = \frac{1}{a'} \times 3118,029$$

$k = 11$

$$(0,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 0,513799$$

$$(1,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -8,49277 \qquad (0,1)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 7,97897$$

$$(2,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 156,8038 \qquad (1,1)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -139,8183 \qquad (0,2)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 123,8604$$

$$(3,0)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -3224,776 \qquad (2,1)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times 2754,365 \qquad (1,2)\ C^{(11)}_{\frac{3}{2}} = \frac{1}{a'} \times -2334,910$$

$k = 12$

$$(0,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 0,385521$$

$$(1,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -6,74764 \qquad (0,1)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 6,36212$$

$$(2,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 130,8682 \qquad (1,1)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -117,3729 \qquad (0,2)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 104,6487$$

$$(3,0)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -2803,076 \qquad (2,1)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 2410,471 \qquad (1,2)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times -2058,352$$

$$(0,3)\ C^{(12)}_{\frac{3}{2}} = \frac{1}{a'} \times 1744,406$$

For the development of the third term,

$k = 10$

$$(0,0)\ C^{(10)}_{\frac{5}{2}} = \frac{1}{a'} \times 10,39741$$

$$(1,0)\ C^{(10)}_{\frac{5}{2}} = \frac{1}{a'} \times -205,5808$$

$k = 11$

$$(0,0)\ C^{(11)}_{\frac{5}{2}} = \frac{1}{a'} \times 8,32969$$

$$(1,0)\ C^{(11)}_{\frac{5}{2}} = \frac{1}{a'} \times -172,3167 \qquad (0,1)\ C^{(11)}_{\frac{5}{2}} = \frac{1}{a'} \times 163,9870$$

48. Employing these numbers in the calculation of $L^{(8)}$, $L^{(9)}$, &c. $M^{(9)}$, $M^{(10)}$, &c. $N^{(10)}$ and $N^{(11)}$, from the expressions in (20), (24), and (28), we obtain the following numerical values:

$$L^{(8)} = \frac{m}{a'} \times -333{,}0969$$

$$L^{(9)} = \frac{m}{a'} \times 1273{,}4929$$

$$L^{(10)} = \frac{m}{a'} \times -1945{,}7913$$

$$L^{(11)} = \frac{m}{a'} \times 1485{,}3152$$

$$L^{(12)} = \frac{m}{a'} \times -566{,}5632$$

$$L^{(13)} = \frac{m}{a'} \times 86{,}3635$$

$$M^{(9)} = \frac{m}{a'} \times -503{,}4795$$

$$M^{(10)} = \frac{m}{a'} \times 1088{,}9148$$

$$M^{(11)} = \frac{m}{a'} \times -787{,}0581$$

$$M^{(12)} = \frac{m}{a'} \times 190{,}0487$$

$$N^{(10)} = \frac{m}{a'} \times -85{,}3347$$

$$N^{(11)} = \frac{m}{a'} \times 58{,}8603$$

49. The computation of these quantities has been effected by means of algebraical operations of great complexity, and numerical calculations of no inconsiderable length; and it is not easy to find in the operations themselves any verification of their accuracy. This has imposed on me the necessity of examining closely every line of figures before I proceeded to another. I have

had the advantage however of comparing the calculated values several times with the values which I calculated nearly four years ago. At that time I developed the principal fraction in a different manner, and I expressed the quantities $C^{(k)}_{\frac{5}{2}}$ &c. by different formulæ; and the fundamental number differed by a few units in the last place of decimals. The numbers admitted of comparison at several intermediate points before arriving at the final results; and one small error was discovered in the old calculations, and one in the new ones. Upon the whole, I am certain that there is no error of importance in these numbers; and I think it highly probable that there is no error, except such as inevitably arise from the rejection of figures beyond a certain place of decimals. It is impossible to assert that the last figure preserved is correct, or even the last but one; but I do not think that the last but two is wrong.

Section 14.

Numerical calculation of the long inequality in the epoch, depending on (13 × mean long. Earth − 8 × mean long. Venus).

50. The most convenient form in which the expression of (29) can be put is the following.

$$\left\{ L^{(8)}.e'^5.\cos(5\varpi') + L^{(9)}.e'^4 e.\cos(4\varpi'+\varpi) + L^{(10)}.e'^3 e^2.\cos(3\varpi'+2\varpi) \right.$$

$$+ L^{(11)}.e'^2 e^3.\cos(2\varpi'+3\varpi) + L^{(12)}.e' e^4.\cos(\varpi'+4\varpi)$$

$$+ L^{(13)}.e^5.\cos(5\varpi) + M^{(9)}.e'^3 f^2.\cos(3\varpi'+2\theta)$$

$$+ M^{(10)} e'^2 e f^2.\cos(2\varpi'+\varpi+2\theta) + M^{(11)}.e' e^2 f^2.\cos(\varpi'+2\varpi+2\theta)$$

$$+ M^{(12)}.e^3 f^2.\cos(3\varpi+2\theta) + N^{(10)}.e' f^4.\cos(\varpi'+4\theta)$$

$$\left. + N^{(11)}.e f^4.\cos(\varpi+4\theta) \right\} \cos\{13(n't+\varepsilon') - 8(nt+\varepsilon)\}$$

$$+ \left\{ L^{(8)}.e'^5.\sin(5\varpi') + L^{(9)}.e'^4 e.\sin(4\varpi'+\varpi) + L^{(10)}.e'^3 e^2.\sin(3\varpi'+2\varpi) \right.$$

$$+ L^{(11)}.e'^2 e^3.\sin(2\varpi'+3\varpi) + L^{(12)}.e' e^4.\sin(\varpi'+4\varpi)$$

$$+ \overset{(13)}{L} . e^5 . \sin (5 \varpi) + \overset{(9)}{M} . e'^3 f^2 . \sin (3 \varpi' + 2 \theta)$$

$$+ \overset{(10)}{M} . e'^2 e f^2 . \sin (2 \varpi' + \varpi + 2 \theta) + \overset{(11)}{M} . e' e^2 f^2 . \sin (\varpi' + 2 \varpi + 2 \theta)$$

$$+ \overset{(12)}{M} . e^3 f^2 . \sin (3 \varpi + 2 \theta) + \overset{(10)}{N} . e' f^4 . \sin (\varpi' + 4 \theta)$$

$$+ \overset{(11)}{N} . e f^4 . \sin (\varpi + 4 \theta) \Big\} . \sin \{13 (n't + \epsilon') - 8 (n t + \epsilon)\}$$

The elements e', e, &c. are all subject to small permanent variation; and (considering the great length of period of the inequality which we are calculating,) those variations may have a sensible influence upon it. It is prudent therefore, as well as interesting, to take into account these variations.

51. Let P and Q be the values of the coefficients of $\cos \{13 (n't + \epsilon') - 8 (nt + \epsilon)\}$ and $\sin \{13 (n't + \epsilon') - 8 (n t + \epsilon)\}$ in the expression above, giving to the elements the values which they had in 1750. Then, as all the permanent variations are small, the powers of t above the first may be rejected, and the coefficients at the time t after 1750 may be represented by $P + p t$ and $Q + q t$. Thus the term of R becomes

$$(P + p t) \cos \{13 (n't + \epsilon') - 8 (n t + \epsilon)\} + (Q + q t) \sin \{13 (n't + \epsilon') - 8 (n t + \epsilon)\};$$

and by (2), omitting the terms depending on $\frac{d R}{d a'}$ and $\frac{d R}{d e'}$ for the reasons in (31),

$$\frac{d n'}{d t} = - \frac{39 n'^2 a'}{\mu'} (P + p t) \sin \{13 (n' t + \epsilon') - 8 (n t + \epsilon)\}$$

$$+ \frac{39 n'^2 a'}{\mu'} (Q + q t) \cos \{13 (n' t + \epsilon') - 8 (n t + \epsilon)\}$$

$$\frac{d \epsilon'}{d t} = + \frac{39 n'^2 a'}{\mu'} (P t + p t^2) \sin \{13 (n' t + \epsilon') - 8 (n t + \epsilon)\}$$

$$- \frac{39 n'^2 a'}{\mu'} (Q t + q t^2) \cos \{13 (n' t + \epsilon') - 8 (n t + \epsilon)\}$$

Integrating these, (considering n', ϵ', n, and ϵ, on the right-hand side, as constants,) and substituting in the expression $n' t + \epsilon'$, it becomes

$$N' t + E'$$

$$+ \frac{39 n'^2 a'}{\mu'} \left\{ \frac{P + p . t}{(13 n' - 8 n)^2} + \frac{2 q}{(13 n' - 8 n)^3} \right\} \sin \{13 (n' t + \epsilon') - 8 (n t + \epsilon)\}$$

$$+\frac{39\,n'^2\,a'}{\mu'}\left\{\frac{-\,Q-q\,t}{(13\,n'-8\,n)^2}+\frac{2\,p}{(13\,n'-8\,n)^3}\right\}\cos\{13\,(n'\,t+\varepsilon')-8\,(n\,t+\varepsilon)\}$$

The terms added to $N'\,t+E'$ constitute the inequality in the epoch.

52. The values of the elements for 1750 and their annual variations are given by LAPLACE in the Mécanique Céleste, 2me Partie, Livre 6, Nos 22 and 26. To give them the form necessary for our purpose, we must from the variation in a Julian year deduce the variation for a unit of time. Now a Julian year is (nearly) the time in which the angle $n'\,t$ increases by $2\,\pi$; its expression is therefore $\frac{2\,\pi}{n'}$. Consequently if we multiply the annual variations by $\frac{n'}{2\,\pi}$, we shall have the variations in a unit of time: and if we multiply them by $\frac{n'\,t}{2\,\pi}$, we shall have the variations in the time t. With regard to the quantities μ', &c. introduced by LAPLACE for the purpose of altering his assumed masses if necessary, it may be observed that the only planet which materially affects the changes of the elements, and whose mass is known with certainty to require a change, is Venus herself. The investigations of BURCKHARDT and BESSEL lead to the same conclusion as my own (Phil. Trans. 1828), namely, that the mass of Venus is $\frac{8}{9}\times$ the mass assumed by DELAMBRE, or $\frac{1}{401211}$ of the sun's mass. LAPLACE supposed it $\frac{1+\mu'}{383137}$ of the sun's mass: the comparison of these gives LAPLACE's $\mu'=-\,{,}045$. In using LAPLACE's expressions, therefore, I shall suppose $\mu'=-\,{,}045$, and $\mu,\ \mu'',\ \mu'''$, &c. $=0$. For convenience, the centesimal* division will be retained.

53. Thus we have

$$n=\frac{650198000}{399993009}\times n'$$

$$e'=0{,}01681395-0{,}0000000729\times n'\,t$$

$$e\ =0{,}00688405-0{,}0000001005\dagger\times n'\,t$$

$$f\ =0{,}02960597+0{,}0000000172\times n'\,t$$

$$\varpi'=109^{g}{,}5790+0{,}0000091017\times n'\,t$$

* BORDA's tables, published by DELAMBRE, have been used in these computations.

† The variations of the elements of Venus do not agree with those of LINDENAU's tables.

$$\varpi = 142^g,1241 - 0,0000018080 \times n' t$$
$$\theta = 82^g,7093 - 0,0000139997 \times n' t$$

The node and inclination are those on the earth's *true* orbit. All the coefficients of $n' t$ are in decimal parts of the radius 1, and not in parts of a degree.

54. From these we deduce the following values, the figures within the brackets being the logarithms of the numbers.

$$e'^5 = +(91,1283485) - (86,46438) \,.\, n' t$$
$$e'^4 e = +(90,7405229) - (86,24488) \,.\, n' t$$
$$e'^3 e^2 = +(90,3526973) - (85,97806) \,.\, n' t$$
$$e'^2 e^3 = +(89,9648717) - (85,68477) \,.\, n' t$$
$$e' e^4 = +(89,5770461) - (85,37453) \,.\, n' t$$
$$e^5 = +(89,1892205) - (85,05252) \,.\, n' t$$
$$e'^3 f^2 = +(91,6197677) - (86,69331) \,.\, n' t$$
$$e'^2 e f^2 = +(91,2319421) - (86,57650) \,.\, n' t$$
$$e' e^2 f^2 = +(90,8441165) - (86,35426) \,.\, n' t$$
$$e^3 f^2 = +(90,4562909) - (86,08606) \,.\, n' t$$
$$e' f^4 = +(92,1111869) - (86,41479) \,.\, n' t$$
$$e f^4 = +(91,7233613) - (86,81239) \,.\, n' t$$

$$\frac{a' L^{(8)}}{m} \,.\, \cos(5\varpi') = +(2,3572098) + (98,04404) \,.\, n' t$$

$$\frac{a' L^{(8)}}{m} \,.\, \sin(5\varpi') = -(2,3859510) + (98,01530) \,.\, n' t$$

$$\frac{a' L^{(9)}}{m} \,.\, \cos(4\varpi' + \varpi) = -(3,0841670) - (98,12469) \,.\, n' t$$

$$\frac{a' L^{(9)}}{m} \,.\, \sin(4\varpi' + \varpi) = +(2,5856285) - (98,62323) \,.\, n' t$$

$$\frac{a' L^{(10)}}{m} \,.\, \cos(3\varpi' + 2\varpi) = +(3,2799989) - (97,97020) \,.\, n' t$$

$$\frac{a' L^{(10)}}{m} \,.\, \sin(3\varpi' + 2\varpi) = +(2,5956493) + (98,65455) \,.\, n' t$$

$$\frac{a' L^{(11)}}{m} \,.\, \cos(2\varpi' + 3\varpi) = -(3,0497482) + (98,09507) \,.\, n' t$$

$$\frac{a' \, L^{(11)}}{m} . \sin (2 \varpi' + 3 \varpi) = - (2{,}9885633) - (98{,}15626) . n' t$$

$$\frac{a' \, L^{(12)}}{m} . \cos (\varpi' + 4 \varpi) = + (2{,}2816808) - (96{,}99874) . n' t$$

$$\frac{a' \, L^{(12)}}{m} . \sin (\varpi' + 4 \varpi) = + (2{,}7269678) + (96{,}55345) . n' t$$

$$\frac{a' \, L^{(13)}}{m} . \cos (5 \varpi) = + (1{,}1565787) - (96{,}88643) . n' t$$

$$\frac{a' \, L^{(13)}}{m} . \sin (5 \varpi) = - (1{,}9302586) - (96{,}11275) . n' t$$

$$\frac{a' M^{(9)}}{m} . \cos (3 \varpi' + 2 \theta) = - (1{,}6642318) - (96{,}54170) . n' t$$

$$\frac{a' M^{(9)}}{m} . \sin (3 \varpi' + 2 \theta) = - (2{,}7001492) + (95{,}50578) . n' t$$

$$\frac{a' M^{(10)}}{m} . \cos (2 \varpi' + \varpi + 2 \theta) = - (2{,}6468283) + (98{,}06223) . n' t$$

$$\frac{a' M^{(10)}}{m} . \sin (2 \varpi' + \varpi + 2 \theta) = + (2{,}9976206) + (97{,}71143) . n' t$$

$$\frac{a' M^{(11)}}{m} . \cos (\varpi' + 2 \varpi + 2 \theta) = + (2{,}8001796) - (98{,}02467) . n' t$$

$$\frac{a' M^{(11)}}{m} . \sin (\varpi' + 2 \varpi + 2 \theta) = - (2{,}6722198) - (98{,}15263) . n' t$$

$$\frac{a' M^{(12)}}{m} . \cos (3 \varpi + 2 \theta) = - (2{,}2752440) + (96{,}91212) . n' t$$

$$\frac{a' M^{(12)}}{m} . \sin (3 \varpi + 2 \theta) = + (1{,}3880761) + (97{,}79929) . n' t$$

$$\frac{a' \, N^{(10)}}{m} . \cos (\varpi' + 4 \theta) = - (1{,}8370062) - (97{,}37538) . n' t$$

$$\frac{a' \, N^{(10)}}{m} . \sin (\varpi' + 4 \theta) = - (1{,}7042256) + (97{,}50816) . n' t$$

$$\frac{a' \mathrm{N}^{(11)}}{m} \cdot \cos(\varpi + 4\,\theta) = +\,(1{,}3847917) + (97{,}49139)\,.\,n'\,t$$

$$\frac{a' \mathrm{N}^{(11)}}{m} \cdot \sin(\varpi + 4\,\theta) = +\,(1{,}7294138) - (97{,}14677)\,.\,n'\,t$$

55. Substituting these in the expressions of (50), we find

$$\mathrm{P} = -\frac{m}{a'} \times (94{,}1302623) \qquad p = +\frac{m}{a'} \times (89{,}08397)\,.\,n'$$

$$\mathrm{Q} = -\frac{m}{a'} \times (94{,}0722348) \qquad q = +\frac{m}{a'} \times (89{,}47976)\,.\,n'$$

and making $\frac{m}{\mu'} = \frac{1}{401211}$, and $13\,n' - 8\,n = -\frac{1674883}{399993090} \times n'$, in the expression of (51), we find for the long inequality

$$\{-(94{,}8787039) + n'\,t \times (89{,}82780)\}\,.\,\sin\{13\,(n'\,t + \varepsilon') - 8\,(n\,t + \varepsilon)\}$$
$$+\,\{+(94{,}8139258) - n'\,t \times (90{,}22359)\}\,.\,\cos\{13\,(n'\,t + \varepsilon') - 8\,(n\,t + \varepsilon)\}$$

which may be put in the form

$$\{+(94{,}9992364) - n'\,t \times (90{,}20461)\}\,.\,\sin\{8\,(n\,t + \varepsilon) - 13\,(n'\,t + \varepsilon')$$
$$+\,40^\circ\,44'\,34'' - n'\,t \times (94{,}91918)\}$$

where the degrees, &c. in the argument are sexagesimal. The coefficient is expressed by a multiple of the radius: to express the principal term in sexagesimal seconds, it must be divided by sin 1″. And if Y be the number of years after 1750, since $n'\,t =$ mean motion of the earth in Y years $= 2\,\pi\,.\,\mathrm{Y} = 6\,.\,60^3\,.\,\mathrm{Y}$ in seconds, the coefficients of $n'\,t$ must be multiplied by $6\,.\,60^3\,.\,\mathrm{Y}$, and their values will then be exhibited in sexagesimal seconds. Thus we find at length for the inequality

$$\{2'',059 - \mathrm{Y} \times 0'',0002076\} \times \sin\{8\,(n\,t + \varepsilon) - 13\,(n'\,t + \varepsilon')$$
$$+\,40^\circ\,44'\,34'' - \mathrm{Y} \times 10'',76\}.$$

56. The mean longitudes $n\,t + \varepsilon$, $n'\,t + \varepsilon'$, are measured from the equinox of 1750. But if l, l', are the mean longitudes of Venus and the Earth measured from the place of the equinox Y years after 1750, then (in consequence of precession)

$$n t + \varepsilon = l - \mathrm{Y} \times 50'',1$$

$$n' t + \varepsilon' = l' - \mathrm{Y} \times 50'',1$$

Consequently $8 (n t + \varepsilon) - 13 (n' t + \varepsilon') = 8 l - 13 l' + \mathrm{Y} \times 250'',5.$

Substituting this, the expression for the inequality is

$$\{2'',059 - \mathrm{Y} \times 0'',0002076\} \times \sin \{8 l - 13 l' + 40° \, 44' \, 34'' + \mathrm{Y} \times 239'',7\}$$

57. I have compared the calculations of the principal part of this inequality with the calculations made in 1827. Two errors were discovered in the former calculations, one of which was important. I am quite confident that there is no sensible error in the results now presented. The terms depending on Y were not calculated on the former occasion: but the calculations now made have been carefully revised.

Section 15.

Numerical calculation of the long inequality in the length of the axis major.

58. This being very small, we shall omit the variable terms. Thus we have

$$\frac{d a'}{d t} = + \frac{26 \, n' a'^2}{\mu'} \mathrm{P} . \sin \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

$$- \frac{26 \, n' a'^2}{\mu} \mathrm{Q} . \cos \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

whence

$$a' = \mathrm{A}' - \frac{26 \, n' a'}{13 \, n' - 8 \, n} \cdot \frac{\mathrm{P} \, a'}{\mu} . \cos \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

$$- \frac{26 \, n' a'}{13 \, n' - 8 \, n} \cdot \frac{\mathrm{Q} \, a'}{\mu'} . \sin \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

$$= \mathrm{A}' - a' . (92,31993) . \cos \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

$$- a' . (92,26190) \sin \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

$$= \mathrm{A}' - a' \times 0,000000027756 \times \cos \{8 (n t + \varepsilon) - 13 (n' t + \varepsilon') + 41° \, 11'\}$$

The magnitude of the coefficient is barely $\frac{1}{40}$th of LAPLACE's minimum, and this inequality may therefore be neglected.

Section 16.

Numerical calculation of the long inequality in the longitude of perihelion.

59. The expression for $\frac{d\varpi'}{dt}$ being $-\frac{n' a'}{\mu' e'} \cdot \frac{dR}{de'}$, the part which we have to consider may be put under the form

$$\begin{aligned}\frac{d\varpi'}{dt} = -\frac{n' a'}{\mu' e'^2} \cdot \Big\{ & 5\,L^{(8)} . e'^5 . \cos(5\varpi') + 4\,L^{(9)} . e'^4 e . \cos(4\varpi' + \varpi) \\ & + 3\,L^{(10)} . e'^3 e^2 . \cos(3\varpi' + 2\varpi) + 2\,L^{(11)} . e'^2 e^3 . \cos(2\varpi' + 3\varpi) \\ & + L^{(12)} . e' e^4 . \cos(\varpi' + 4\varpi) + 3\,M^{(9)} . e'^3 f^2 . \cos(3\varpi' + 2\theta) \\ & + 2\,M^{(10)} . e'^2 e f^2 . \cos(2\varpi' + \varpi + 2\theta) + M^{(11)} . e' e^2 f^2 . \cos(\varpi' + 2\varpi + 2\theta) \\ & + N^{(10)} . e' f^4 . \cos(\varpi' + 4\theta) \Big\} \cos\Big\{ 13(n't + \varepsilon') - 8(nt + \varepsilon) \Big\} \\ -\frac{n' a'}{\mu' e'^2} \Big\{ & 5\,L^{(8)} . e'^5 . \sin(5\varpi') + 4\,L^{(9)} . e'^4 e . \sin(4\varpi' + \varpi) \\ & + 3\,L^{(10)} . e'^3 e^2 . \sin(3\varpi' + 2\varpi) + 2\,L^{(11)} . e'^2 e^3 . \sin(2\varpi' + 3\varpi) \\ & + L^{(12)} . e' e^4 . \sin(\varpi' + 4\varpi) + 3\,M^{(9)} . e'^3 f^2 \sin(3\varpi' + 2\theta) \\ & + 2\,M^{(10)} . e'^2 e f^2 . \sin(2\varpi' + \varpi + 2\theta) + M^{(11)} . e' e^2 f^2 . \sin(\varpi' + 2\varpi + 2\theta) \\ & + N^{(10)} . e' f^4 . \sin(\varpi' + 4\theta) \Big\} \sin\Big\{ 13(n't + \varepsilon') - 8(nt + \varepsilon) \Big\}\end{aligned}$$

which (neglecting the variable terms) is found to equal

$$\begin{aligned}& n' \times (92{,}35866) . \cos\{13(n't + \varepsilon') - 8(nt + \varepsilon)\} \\ & \qquad + n' \times (92{,}60190) . \sin\{13(n't + \varepsilon') - 8(nt + \varepsilon)\}\end{aligned}$$

Integrating,

$$\begin{aligned}\varpi' = \Pi' & - (94{,}73673) . \sin\{13(n't + \varepsilon') - 8(nt + \varepsilon)\} \\ & + (94{,}97997) . \cos\{13(n't + \varepsilon') - 8(nt + \varepsilon)\}\end{aligned}$$

or

$$\varpi' = \Pi' + 1'',1250 . \sin\{8(nt + \varepsilon) - 13(n't + \varepsilon')\}$$

$$+ 1'',9697 . \cos \{8 (n t + \varepsilon) - 13 (n' t + \varepsilon')\}$$

$$= \Pi' + 2'',2683 . \sin \{8 (n t + \varepsilon) - 13 (n' t + \varepsilon') + 60° 16'\}$$

Section 17.

Numerical calculation of the long inequality in the excentricity.

60. On forming the expression for $\frac{d e'}{d t}$, or $+ \frac{n' a'}{\mu' e'} . \frac{d R}{d \varpi'}$, it is immediately seen that the coefficients of $\cos \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$ and $\sin \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$ are related to those above, and that

$$\frac{d e'}{d t} = + e' n' \times (92,60190) . \cos \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$
$$- e' n' \times (92,35866) . \sin \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

Integrating,

$$e' = E' - e' \times (94,97997) . \sin \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$
$$- e' \times (94,73673) . \cos \{13 (n' t + \varepsilon') - 8 (n t + \varepsilon)\}$$

$$= E' - (92,96240) \cos \{8 (n t + \varepsilon) - 13 (n' t + \varepsilon')\}$$
$$+ (93,20564) . \sin \{8 (n t + \varepsilon) - 13 (n' t + \varepsilon')\}$$

$$= E' - 0,0000001849 . \cos \{8 (n t + \varepsilon) - 13 (n' t + \varepsilon') + 60° 16'\}$$

The principal inequality in the radius vector is that produced by the last term: it is however too small to be sensible.

PART II.

PERTURBATION OF THE EARTH IN LATITUDE.

Section 18.

Explanation of the method used here.

61. If ϕ' be the inclination of the earth's orbit to the plane of $x y$, and θ' the longitude of the node, then

$$\frac{d\theta'}{dt} = -\frac{n'a'}{\mu'\sqrt{1-e'^2}} \cdot \frac{1}{\phi'} \cdot \frac{dR}{d\phi'}$$

$$\frac{d\phi'}{dt} = +\frac{n'a'}{\mu'\sqrt{1-e'^2}} \cdot \frac{1}{\phi'} \cdot \frac{dR}{d\theta'}$$

or, neglecting e'^2,

$$\frac{d\theta'}{dt} = -\frac{n'a'}{\mu'} \cdot \frac{1}{\phi'} \cdot \frac{dR}{d\phi'}$$

$$\frac{d\phi'}{dt} = +\frac{n'a'}{\mu'} \cdot \frac{1}{\phi'} \cdot \frac{dR}{d\theta'}$$

These expressions are true only when ϕ' is so small that its square may be rejected. This restriction, however, is convenient as well as necessary. For in the expansion of R we shall have to proceed only to the first power of ϕ', and make $\phi' = 0$ when we have arrived at our ultimate result: consequently the same values of θ and ϕ must be employed as in the first Part.

62. The only term of R, which by expansion will produce terms of the form cos (13 — 8) with coefficients of the fifth order, is the fraction $\frac{-m}{\sqrt{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}}}$. For substitution in the denominator we have

$$x' = r'\cos v' \text{ (neglecting } \phi^2)$$

$$y' = r'\sin v'$$

$$z' = r'.\phi'.\sin(v'-\theta')$$

$$x = r\{\cos(v-\theta).\cos\theta - \cos\phi.\sin(v-\theta).\sin\theta\}$$

$$y = r\{\cos(v-\theta).\sin\theta + \cos\phi.\sin(v-\theta).\cos\theta\}$$

$$z = r.\sin\phi.\sin(v-\theta)$$

whence the fraction is changed to

$$\frac{-m}{\sqrt{\{r'^2-2r'r.\cos(v'-v)+r^2+2r'r.f^2.\cos(v'-v)-2r'r.f^2.\cos(v'+v-2\theta)-4r'r.\phi'f.\sin(v'-\theta').\sin(v-\theta)\}}}$$

where f is put for $\sin\frac{\phi}{2}$ and $2f$ for $\sin\phi$, on the principle of (13). The part of this depending on the first power of ϕ' is

$$\frac{-m.2r'r.\phi'f.\sin(v'-\theta').\sin(v-\theta)}{\{r'^2-2r'r.\cos(v'-v)+r^2+2r'r.f^2.\cos(v'-v)-2r'r.f^2.\cos(v'+v-2\theta\}^{\frac{3}{2}}}$$

of which, on the principle of (8), &c., we are to take only

$$\frac{m \, . \, r' \, r \, . \, \phi' f \, . \cos (v' + v - \theta' - \theta)}{\{r'^2 - 2 \, r' \, r \, . \cos (v' - v) + r^2 - 2 \, r' \, r \, . f^2 \, . \cos (v' + v - 2 \, \theta)\}^{\frac{3}{2}}}$$

Expanding the denominator by powers of f^2, this becomes

$$\frac{m \, . \, r' \, r \, . \, \phi' f \, . \cos (v' + v - \theta' - \theta)}{\{r'^2 - 2 \, r' \, r \, . \cos (v' - v) + r^2\}^{\frac{3}{2}}} + \frac{m \, . \, 3 \, r'^2 \, r^2 \, . \, \phi' f^3 \, . \cos (v' + v - \theta' - \theta) \, . \cos (v' + v - 2 \, \theta)}{\{r'^2 - 2 \, r' \, r \, . \cos (v' - v) + r^2\}^{\frac{5}{2}}}$$

or

$$\frac{m \, . \, r' \, r \, . \, \phi' f \, . \cos (v' + v - \theta' - \theta)}{\{r'^2 - 2 \, r' \, r \, . \cos (v' - v) + r^2\}^{\frac{3}{2}}} + \frac{3}{2} \, . \, \frac{m \, . \, r'^2 \, r^2 \, . \, \phi' f^3 \, . \cos (2 \, v' + 2 \, v - \theta' - 3 \, \theta)}{\{r'^2 - 2 \, r' \, r \, . \cos (v' - v) + r^2\}^{\frac{5}{2}}}$$

Section 19.

Selection of the coefficients of cos (13 − 8) *in the development of the two last fractions.*

63. If we compare the first fraction with the fraction developed in Section 8, we perceive that the following are the only differences between them. The signs of the coefficients are different: and in the coefficient of the new fraction (and in every term of its development) there is ϕ' instead of f, with the corresponding change of argument. From this it is readily seen that the coefficient of cos (13 − 8) will be formed from that in Section 8 (Art. 24), by changing the sign and multiplying by $\frac{\phi'}{f}$; the argument always being changed according to the rules of (9). The coefficient is therefore

$$- \overset{(9)}{\mathrm{M}} \, . \, e'^3 \, \phi' f - \overset{(10)}{\mathrm{M}} \, . \, e'^2 \, e \, \phi' f - \overset{(11)}{\mathrm{M}} \, . \, e' \, e^2 \, \phi' f - \overset{(12)}{\mathrm{M}} \, . \, e^3 \, \phi' f.$$

64. If we compare the second fraction with the fraction developed in Section 10, we see that there are the same differences as those mentioned above, with this additional one, that the multiplier is double of the multiplier of the fraction in Section 10. Thus the coefficient of cos (13 − 8) is found to be

$$- 2 \, \overset{(10)}{\mathrm{N}} \, . \, e' \, \phi' f^3 - 2 \, \overset{(11)}{\mathrm{N}} \, . \, e \, \phi' f^3$$

The sum of the terms in these two sets, multiplied respectively by the cosines of their proper arguments, constitutes the whole term of R which we have to consider.

Section 20.

Numerical calculation of the perturbation in latitude.

65. The first of the terms found in the last section is $-\overset{(9)}{M}.e'^3\varphi' f.\cos\{13(n't+\varepsilon')-8(nt+\varepsilon)-3\varpi'-\theta'-\theta\}$. With respect to this term only, $\frac{dR}{d\varphi'}=-\overset{(9)}{M}.e'^3 f.\cos\{13(n't+\varepsilon')-8(nt+\varepsilon)-3\varpi'-\theta'-\theta\}$; whence

$$\theta'=\Theta'-\frac{n'a'}{\mu'\Phi'}\int_t\frac{dR}{d\Phi'}=\Theta'+\frac{\overset{(9)}{M}a'}{\mu'}.\frac{n'}{13n'-8n}.\frac{e'^3f}{\Phi'}\sin\{13(n't+\varepsilon')$$
$$-8(nt+\varepsilon)-3\varpi'-\theta'-\theta\}.$$

And

$$\frac{dR}{d\theta'}=-\overset{(9)}{M}.e'^3\varphi' f.\sin\{13(n't+\varepsilon)-8(nt+\varepsilon)-3\varpi'-\theta'-\theta\};$$

whence

$$\varphi'=\Phi'+\frac{n'a'}{\mu'\Phi'}\int_t\frac{dR}{d\theta'}=\Phi'+\frac{\overset{(9)}{M}a'}{\mu'}.\frac{n'}{13n'-8n}e'^3f.\cos\{13(n't+\varepsilon')$$
$$-8(nt+\varepsilon)-3\varpi'-\theta'-\theta\}.$$

The Earth's latitude, neglecting small terms, is $\varphi'.\sin(n't+\varepsilon'-\theta')$. And from the expression above, $\sin(n't+\varepsilon'-\theta')=$

$$\sin(n't+\varepsilon'-\Theta')-\frac{\overset{(9)}{M}a'}{\mu'}.\frac{n'}{13n'-8n}.\frac{e'^3f}{\Phi'}.\cos(n't+\varepsilon'-\Theta').\sin$$
$$\{13(n't+\varepsilon')-8(nt+\varepsilon)-3\varpi'-\theta'-\theta\}$$

Multiplying this by the expression for φ', and putting θ', φ', for Θ', Φ', in the small terms, we find for the latitude

$$\Phi'.\sin(n't+\varepsilon'-\Theta')-\frac{\overset{(9)}{M}a'}{\mu}.\frac{n'}{13n'-8n}.e'^3f.\sin\{12(n't+\varepsilon')$$
$$-8(nt+\varepsilon)-3\varpi'-\theta\}$$

and the last part, or the perturbation in latitude, is

$$-\frac{n'}{13n'-8n}.e'^3f.\frac{\overset{(9)}{M}a'}{\mu}.\sin\{12(n't+\varepsilon')-8(nt+\varepsilon)-3\varpi'-\theta\}$$

Similar expressions will be obtained from all the other terms.

66. If we put for $\sin\{12(n't+\varepsilon')-8(nt+\varepsilon)-3\varpi'-\theta\}$ its equivalent $\cos(3\varpi'+2\theta).\sin\{12(n't+\varepsilon')-8(nt+\varepsilon)+\theta\}-\sin(3\varpi'+2\theta).\cos\{12(n't+\varepsilon')-8(nt+\varepsilon)+\theta\}$, and similarly for the other terms, we find for the whole coefficient of $\sin\{12(n't+\varepsilon')-8(nt+\varepsilon)+\theta)\}$,

$$-\frac{n'}{13n'-8n}\cdot\frac{m}{\mu}\cdot\frac{1}{f}\cdot\frac{a'}{m}\Big\{e'^3f^2.\overset{(9)}{\mathrm{M}}.\cos(3\varpi'+2\theta)$$
$$+e'^2ef^2.\overset{(10)}{\mathrm{M}}.\cos(2\varpi'+\varpi+2\theta)+e'e^2f^2.\overset{(11)}{\mathrm{M}}.\cos(\varpi'+2\varpi+2\theta)$$
$$+e^3f^2.\overset{(12)}{\mathrm{M}}.\cos(3\varpi+2\theta)+2e'f^4.\overset{(10)}{\mathrm{N}}.\cos(\varpi'+4\theta)$$
$$+2ef^4.\overset{(11)}{\mathrm{N}}.\cos(\varpi+4\theta)\Big\}$$

and for the whole coefficient of $\cos\{12(n't+\varepsilon')-8(nt+\varepsilon)+\theta\}$,

$$+\frac{n'}{13n'-8n}\cdot\frac{m}{\mu}\cdot\frac{1}{f}\cdot\frac{a'}{m}\Big\{e'^3f^2.\overset{(9)}{\mathrm{M}}.\sin(3\varpi'+2\theta)$$
$$+e'^2ef^2.\overset{(10)}{\mathrm{M}}.\sin(2\varpi'+\varpi+2\theta)+e'e^2f^2.\overset{(11)}{\mathrm{M}}.\sin(\varpi'+2\varpi+2\theta)$$
$$+e^3f^2.\overset{(12)}{\mathrm{M}}.\sin(3\varpi+2\theta)+2e'f^4.\overset{(10)}{\mathrm{N}}.\sin(\varpi'+4\theta)$$
$$+2ef^4.\overset{(11)}{\mathrm{N}}.\sin(\varpi+4\theta)\Big\}$$

On performing the calculations, the inequality is found to be

$$+0'',0086.\sin\{8(nt+\varepsilon)-12(n't+\varepsilon')-\theta\}$$
$$+0'',0060.\cos\{8(nt+\varepsilon)-12(n't+\varepsilon')-\theta\}$$

or $\quad +0'',0105.\sin\{8(nt+\varepsilon)-12(n't+\varepsilon')-39^\circ\,29'\}$

which is too small to be sensible in any observations.

PART III.

PERTURBATIONS OF VENUS DEPENDING ON THE SAME ARGUMENTS.

67. If we consider Venus as disturbed by the Earth, and take the orbit of Venus for the plane xy, the term involving cos (13 − 8) in the expansion of R

will be exactly the same as when we consider the Earth disturbed by Venus. For the longitudes of perihelia and the longitude of the node will be the same: the sign of f will be different, but as the even powers only of this quantity enter into the expansion of R, and as its magnitude (without respect to its sign) is the same in both, that circumstance makes no difference. It is only necessary then to put m' instead of m in the multiplier of the term.

68. First, then, for the inequality in the epoch. Observing that in the expression of (51) the multiplier m is included in P, p, Q, and q, it will be seen that for the perturbation of Venus we must use the multiplier $-\frac{24\,n^2\,a\,m'}{\mu}$ instead of $\frac{39\,n'^2\,a'\,m}{\mu'}$. That is, the argument of the perturbation of Venus is the same as that of the Earth; and its coefficient is found by multiplying the coefficient for the Earth by $-\frac{8\,n^2\,a\,m'}{13\,n'^2\,a'\,m}$. As $8\,n = 13\,n'$ nearly, this fraction $= -\frac{n}{n'}\cdot\frac{a}{a'}\cdot\frac{m'}{m} = -\frac{13}{8}\cdot\frac{a}{a'}\cdot\frac{m'}{m}$ nearly. Assuming $\frac{m'}{\mu} = \frac{1}{329630}$, and the other quantities as before, this fraction is $-\frac{13}{8}\times 0{,}72333\times\frac{401211}{329630}$: whence the long inequality in the epoch of Venus =

$$\{-2'',946 + \text{Y}\times 0'',0002970\}\times\sin\{8\,l - 13\,l' + 40^\circ\,44'\,34'' + \text{Y}\times 239'',7\}$$
$$= \{2'',946 - \text{Y}\times 0'',0002970\}\times\sin\{8\,l - 13\,l' + 220^\circ\,44'\,34'' + \text{Y}\times 239'',7\}$$

The corresponding inequality in the axis major, like that for the Earth, is insensible.

69. For the long inequality in the longitude of perihelion. This cannot be deduced from that of the Earth: but, calculating it independently in the same manner, it is found that

$$\varpi = \Pi - 0'',008\sin(8\,l - 13\,l') - 5'',704\,.\cos(8\,l - 13\,l')$$

70. For the long inequality in the excentricity. This may be derived from that in the longitude of perihelion in the same manner in which it was done for the Earth: thus it appears that

$$e = \text{E} - 0{,}0000001904\,.\sin(8\,l - 13\,l') + 0{,}0000000003\,.\cos(8\,l - 13\,l')$$

71. For the inequality in latitude. The orbit of Venus must now be sup-

posed to be inclined at a small angle to the plane of $x\,y$. We have remarked that, in the development of R for Venus as the disturbed body, the sign of f will be changed: and as the term of R on which the perturbation in latitude depends is a multiple of odd powers of f, the sign for Venus will be different from that for the Earth. Besides this there will be no difference, except that $a\,m'\,n$ is to be substituted for $a'\,m\,n'$. Proceeding then as in (65), and considering the effect of the first term of (63), we find

$$\theta = \Theta - \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\mathrm{M}^{(9)} a'}{m} \cdot \frac{e'^3 f}{\varphi} . \sin \{13\,(n't + \varepsilon') - 8\,(n\,t + \varepsilon) - 3\,\varpi' - \theta' - \theta\}$$

whence

$$\sin(nt + \varepsilon - \theta) = \sin(n\,t + \varepsilon - \Theta) + \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\mathrm{M}^{(9)} a'}{m} \cdot \frac{e'^3 f}{\Phi} \times \cos(n\,t + \varepsilon - \theta) . \sin \{13\,(n'\,t + \varepsilon') - 8\,(n\,t + \varepsilon) - 3\,\varpi' - \theta' - \theta\}$$

And

$$\varphi = \Phi - \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\mathrm{M}^{(9)} a'}{m} \cdot e'^3 f . \cos \{13\,(n'\,t + \varepsilon') - 8\,(n\,t + \varepsilon) - 3\,\varpi' - \theta' - \theta\}$$

The product of these expressions gives for the latitude of Venus

$$\Phi . \sin(n\,t + \varepsilon - \Theta) + \frac{n}{13\,n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\mathrm{M}^{(9)} a'}{m} . e'^3 f . \sin \{13\,(n'\,t + \varepsilon') - 9\,(n\,t + \varepsilon) - 3\,\varpi' - \theta'\}$$

where θ' has the same value which θ had in the investigation for the Earth. The perturbation in latitude is therefore $\frac{n}{13n' - 8\,n} \cdot \frac{m'}{\mu} \cdot \frac{a}{a'} \cdot \frac{\mathrm{M}^{(9)} a'}{m} e'^3 f . \sin \{13\,(n'\,t + \varepsilon') - 9\,(n\,t + \varepsilon) - 3\,\varpi' - \theta'\}$, and similarly for the other terms. Comparing this with the term in (65) it will readily be seen that we have only to multiply the expression of (66) by $- \frac{m'\,n\,a}{m\,n'\,a'}$, and to put $9\,(n\,t + \varepsilon) - 13\,(n'\,t + \varepsilon')$ instead of $8\,(n\,t + \varepsilon) - 12\,(n'\,t + \varepsilon')$, and the perturbation of Venus in latitude will be found. Thus it becomes

$$-0'',0123 \,.\, \sin\{9(nt+\varepsilon)-13(n't+\varepsilon')-\theta\}-0'',0086 \,.\, \cos\{9(nt+\varepsilon) - 13(n't+\varepsilon')-\theta\}$$

or

$$+0'',0151 \times \sin\{9(nt+\varepsilon)-13(n't+\varepsilon')+140^\circ\ 31'\}$$

which, though larger than the Earth's perturbation in latitude, is too small to be observable.

Conclusion.

It appears, then, that in calculating the Earth's longitude (or 180° + Sun's longitude), the following terms should be used in addition to those that have hitherto been applied; (where l and l' are the mean tropical longitudes of Venus and the Earth, and Y the number of years after 1750:)

To the epoch of mean longitude

$$+\{2'',059-\mathrm{Y}\times 0'',0002076\}\times\sin\{8\,l-13\,l'+40^\circ\ 44'\ 34''+\mathrm{Y}\times 239'',7\}$$

To the epoch of longitude of perihelion

$$+2'',268\times\sin\{8\,l-13\,l'+60^\circ\ 16'\}$$

To the excentricity

$$-0,0000001849 \,.\, \cos\{8\,l-13\,l'+60^\circ\ 16'\}$$

and that, in calculating the Earth's latitude (or the Sun's latitude with sign changed), the following term should be used;

$$+0'',0105 \,.\, \sin\{8\,l-12\,l'-39^\circ\ 29'\}$$

Similarly, it appears that in calculating the place of Venus, the following terms should be applied:

To the epoch of mean longitude

$$+\{2'',946-\mathrm{Y}\times 0'',0002970\}\times\sin\{8\,l-13\,l'+220^\circ\ 44'\ 34''+\mathrm{Y}\times 239'',7\}$$

To the longitude of perihelion

$$-5'',70 \,.\, \cos\{8l-13l'\}$$

To the excentricity

$$-0,000000190 \,.\, \sin\{8\,l-13\,l'\}$$

To the latitude

$$+0'',0151 \,.\, \sin\{9\,l-13\,l'+140^\circ\ 31'\}$$

The terms affecting the latitude may be at once neglected. The inequalities in longitude produced by the change of mean anomaly and excentricity, ($n\,t + \varepsilon - \varpi$ and e), and which are

for the Earth

$$-0'',0470 \times \sin\{8\,l - 12\,l' - 15^\circ\ 34'\} - 0'',0346 \,.\, \sin\{14\,l' - 8\,l - 139^\circ\ 22'\}$$

for Venus

$$+0'',0671 \,.\, \sin\{9\,l - 13\,l' - 24^\circ\ 40'\} + 0'',0203 \,.\, \sin\{13\,l' - 7\,l - 168^\circ\ 40'\}$$

can scarcely be detected from observation. The inequalities in the radii vectores are not sensible.

The long inequalities in the epoch of longitude are however by no means to be neglected. To point out a single instance in which their importance will be sensible, I will estimate roughly their effect on the places of the Earth and Venus at the next transit of Venus over the Sun's disk (in 1874). The value of these inequalities at the time of BRADLEY's observations was small; and they were at their maximum at the beginning of this century. If, then, the mean motions of the Earth and Venus were determined by comparing the observations about BRADLEY's time with the observations a few years ago; the Earth's longitude in 1874, when the inequalities are nearly vanishing, would be too small by nearly 4″; that of Venus would be too great by 6″: their difference of longitude would therefore be nearly 10″ in error; and this would produce on the geocentric* longitude of Venus an effect of between 20″ and 30″. As another instance, I may mention that the secular motions of the Earth, determined from observations of two consecutive centuries, would differ nearly 8″, and those of Venus nearly 12″.

These inequalities vanish in the years 1622, 1742, and 1861; and have their greatest values, positive for the Earth and negative for Venus, in 1682; and negative for the Earth and positive for Venus, in 1802. At the principal transits of Venus their values are as follows:

* In the Memoirs of the Astronomical Society I have pointed out the utility of observations of Venus near inferior conjunction for determining the coefficient of the inequality in the Earth's motion, produced by the Moon. I take this opportunity of repeating my conviction, that observations of Venus near inferior conjunction are adapted better than any others to the detection and measurement of minute inequalities in the Earth's motion.

	For the Earth.	For Venus.
In 1639	$+0'',89$	$-1'',28$
1761	$-0,98$	$+1,41$
1769	$-1,34$	$+1,91$
1874	$+0,68$	$-0,97$
1882	$+1,07$	$-1,53$

I shall now show the coincidence of the theoretical results with the observations that first suggested their necessity.

From BURCKHARDT's examination of MASKELYNE's observations (Connaissance des Temps, 1816), and from my examination of Mr. POND's observations (Phil. Trans. 1828), it appeared that the mean longitudes of DELAMBRE's tables ought to be increased

in 1783 by $0'',25$
in 1801 by $0,08$
in 1821 by $2,05$

These observations are all reduced by the same catalogue. The differences of the corrections are not proportional to the intervals; and this is the circumstance that shows the existence of some periodical inequality.

Now the values of the argument of the long inequality in the epoch are

for 1783	240° 59′
for 1801	268 4
for the middle of 1821	298 46

The sines of these angles are $-0,8745$, $-0,9994$, $-0,8766$; and hence the values of the inequality were

in 1783	$-1'',80$
in 1801	$-2,06$
in 1821	$-1,81$

If these had been applied in the tables, the corrections given by the observations above would have been

in 1783	$2'',05$
in 1801	$2,86$
in 1821	$3,86$

and the differences between these are almost exactly proportional to the times. They show that the secular motion ought to be increased by $4'',8$ (the precession being supposed the same as in the application of MASKELYNE's catalogue); and then the application of the inequality investigated in this memoir will give correctly the Sun's mean longitude.

It appears, however, that the inequality in the motion of the perihelion given by this investigation, will not account for the anomalies in the place of the perihelion given in my paper referred to above.

Thus terminates one of the most laborious investigations that has yet been made in the Planetary Theory. The term in question is a striking instance of the importance to which terms, apparently the most insignificant, may sometimes rise; and the following remark will show the magnitude of the errors which might, under other circumstances, have arisen from the neglect of this term. If the perihelia of Venus and the Earth had opposite longitudes, and if the line of nodes coincided with the major axes, the excentricities and inclination having the same values as at present, the coefficient of the inequality in the epoch would be $8'',9$; and all the other terms would be important. A very small increase of the excentricities and inclination would double or treble these inequalities.

I have avoided any discussion of physical theory, as little can be added at present to what has been done by LAPLACE and others. I may remark, however, that my expression for $\frac{d\epsilon}{dt}$ differs from that given by LAPLACE; and that the difference produces no effect in the ultimate result, because LAPLACE uses $\int n\,dt$ where I have used nt. On this point I have only to state that, by adopting the expression which I have used, every formula for the longitude, the radius vector, and the velocity in any direction, is exactly the same in form for the variable ellipse as for an invariable ellipse (taking the variable elements instead of constant ones). If the disturbing force should at any instant cease, my value of ϵ for that instant would be the true value of the epoch of mean longitude in the orbit which the planet would proceed to describe. It is precisely the object of using the method of variation of elements, to obtain expressions which possess these properties; and therefore I have little doubt that my form will be recognised as more completely in accordance with the principles of that method than LAPLACE's. I should not in the present instance have raised a question on this point, but that I conceive the method of variation of elements, or some similar method, possessing the same advantages of simplicity of application and unlimited accuracy as to the order of the disturbing force, will ere long be adopted in the Planetary Theories, to the total exclusion of other methods. With this expectation, it appears important to adhere closely to the principles of the theory in every formula that is derived from it.

I believe that the paper now presented to the Royal Society contains the first* specific improvement in the Solar Tables made in this country since the establishment of the Theory of Gravitation. And I have great pleasure in reflecting that, after having announced a difficulty detected by observation, I have been able to offer an explanation on the grounds of physical theory.

POSTSCRIPT.

In estimating the variation of the elements of the orbit of Venus, the change of longitude of perihelion was supposed to be the same as the sidereal motion of the perihelion. This is not strictly true; as the longitude of the perihelion, measured as in Art. 4, depends upon the place of the node, and is affected therefore by the motion of the node as well as by the motion of the perihelion. The amount of the error is however perfectly insignificant.

G. B. AIRY.

Observatory, Cambridge,
Nov. 8, 1831.

* I am not aware that anything has been added to the theory of planetary perturbation, by an Englishman, from the publication of NEWTON's Principia to the communication of Mr. LUBBOCK's Researches. In MASKELYNE's tables are two for the perturbations of the Earth produced by Venus and Jupiter, calculated (he states) by himself; but they are utterly useless and erroneous, as they contain no terms depending on the excentricities.